



NUMBER SYSTEMS

Natural Numbers:

All counting numbers are natural numbers. We denote this collection by the symbol N .

e.g. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, ...

Whole Numbers:

Natural numbers together with 0 are called whole numbers which are denoted by the symbol W .

e.g. 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, ...

Integers:

Integers are a collection of whole numbers with negative numbers, and it is denoted by the symbol Z .

e.g. -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, ...

It can be divided into positive, negative and zero.

Positive integers e.g. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ... and

Negative integers e.g. -1, -2, -3, -4, -5, -6, -7, -8, -9, -10, ...

NOTE: 0 (zero) is neither negative nor positive.

Rational Numbers:

A number r is called a rational number, if it can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. The collection of rational numbers is denoted by Q .

e.g. $1, \frac{2}{3}, \frac{-3}{4}, \frac{-5}{2}, 0, -2, 1.5, 2.\overline{54}, \dots$

NOTE: There are infinitely many rational numbers between any two given rational numbers.

Irrational Numbers:

A number s is called an irrational number, if it cannot be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

e.g. $\sqrt{2}, \sqrt{5}, -\sqrt{3}, 1.53243434003567889..., \sqrt{\frac{2}{3}}, \dots$

Real Numbers:

All rational numbers and irrational numbers together make up the collection of real numbers, which is denoted by $\mathbf{R}$. Therefore, a real number is either rational or irrational.

e.g. $1, \frac{2}{3}, \frac{-3}{4}, \frac{-5}{2}, 0, -2, 1.5, 2.\overline{54}, \sqrt{2}, \sqrt{5}, -\sqrt{3}, 1.53243434003567889..., \sqrt{\frac{2}{3}}, \dots$

Real Numbers and their Decimal Expansions:

The decimal expansion of a rational number is either terminating or non-terminating recurring. In other words, a number whose decimal expansion is terminating, or non-terminating recurring is rational.

e.g. $1.5, 0.59, 1.\overline{3}, 3.50\overline{4}, 1.\overline{2345}, \dots$

Let us consider a few examples:

Example: Show that $1.\overline{3}$ is a rational number.

Solution: We can write $1.\overline{3} = 1.333\dots$

So, let $x = 1.333\dots$ (i)

Now, multiplying equation (i) by 10 as one-digit is recurring after decimal

$$10 \times x = 10 \times (1.333\dots)$$

$$10x = 13.333\dots$$

$$10x = 12 + 1.333\dots$$

By using equation (i), we get:

$$10x = 12 + x$$

$$10x - x = 12$$

$$9x = 12$$

$$x = \frac{12}{9}$$

$$x = \frac{4}{3}$$

Therefore, $1.\bar{3} = \frac{4}{3}$

Hence $1.\bar{3}$ is a rational number.

Example: Express $3.2\overline{45}$ in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$ also p and q are co-primes.

Solution: Let $x = 3.2\overline{45}$

Now, multiplying both sides by 10 to separate repeating decimal place

$$10x = 32.\overline{45}$$

$$10x = 32.454545... \quad (i)$$

Now, multiplying equation (i) by 100 as two-digits are recurring after decimal

$$100 \times 10x = 100 \times 32.454545...$$

$$1000x = 3245.454545...$$

$$1000x = 3213 + 32.454545...$$

By using equation (i), we get:

$$1000x = 3213 + 10x$$

$$1000x - 10x = 3213$$

$$990x = 3213$$

$$x = \frac{3213}{990}$$

$$x = \frac{357}{110}$$

Therefore, $3.2\overline{45} = \frac{357}{110}$

The decimal expansion of an irrational number is non-terminating non-recurring. In other words, a number whose decimal expansion is non-terminating non-recurring is irrational.

$$\sqrt{2} = 1.4142135623730950488016887242096\dots$$

e.g. $\pi = 3.14159265358979323846264338327950\dots$

Representing Real Numbers on the Number Line:

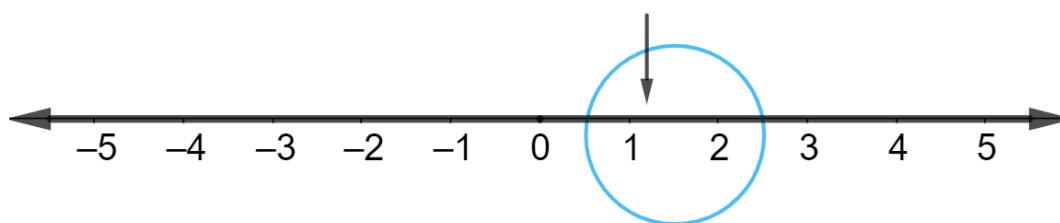
The process of visualization of representation of numbers on the number line, through a magnifying glass, is called the **process of successive magnification**.

Let us consider a few examples:

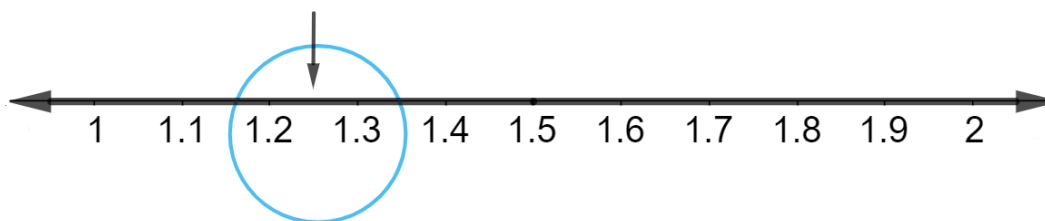
Example: Represent 1.256 on a number line

Solution: We know that 1.256 lies between whole numbers 1 and 2.

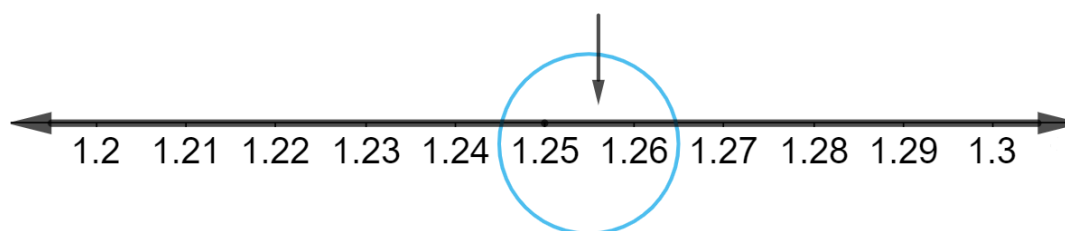
So, let us look closely at the portion of the number line between 1 and 2. Suppose we divide this into 10 equal parts and mark each point of division as:



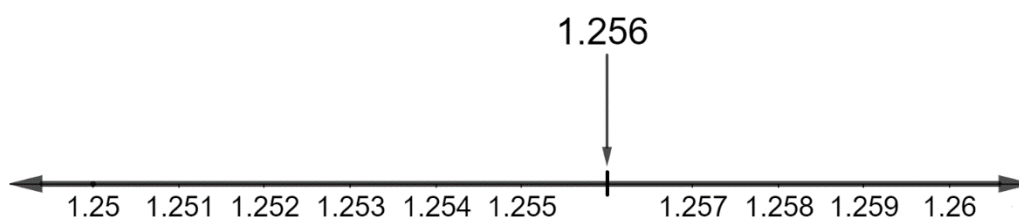
Then the first mark to the right of 1 will represent 1.1, the second 1.2, and so on. We might be finding some difficulty in observing these points of division between 1 and 2. To have a clear view of the same, we may take a magnifying glass and look at the portion between 1 and 2. It will look like this:



Now 1.256 lies between 1.2 and 1.3. So, let us focus on the portion between 1.2 and 1.3. We imagine dividing this again into ten equal parts. The first mark will represent 1.21, the next 1.22, and so on. To see this clearly, we magnify this as shown:



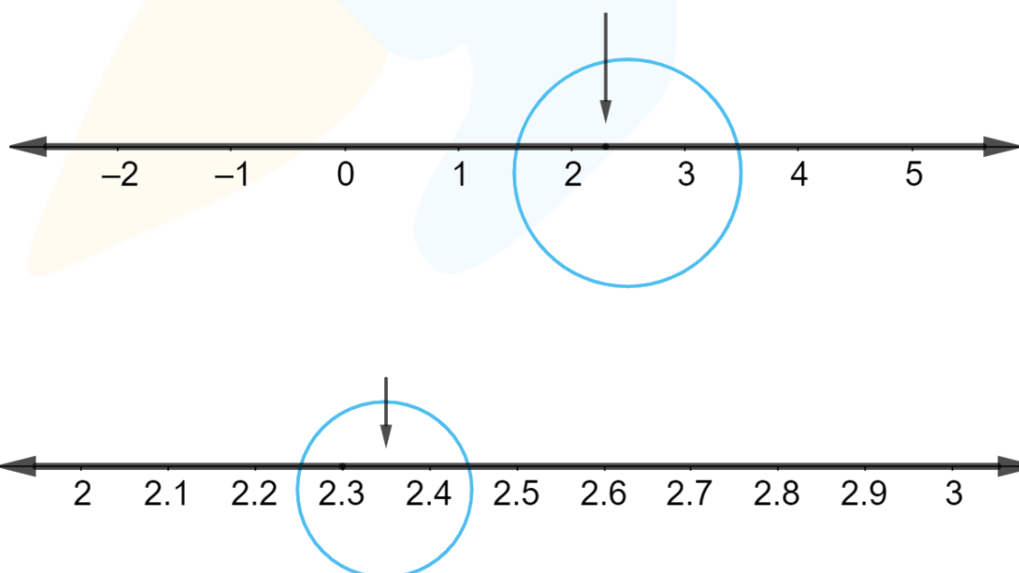
Again, 1.256 lies between 1.25 and 1.27. So, let us focus on this portion 1.25 and 1.26 of the number line and imagine dividing it again into ten equal parts. The first mark represents 1.251, the next one represents 1.252, and so on. So, 1.256 is the 6th mark in these subdivisions. We magnify it to see it better as:

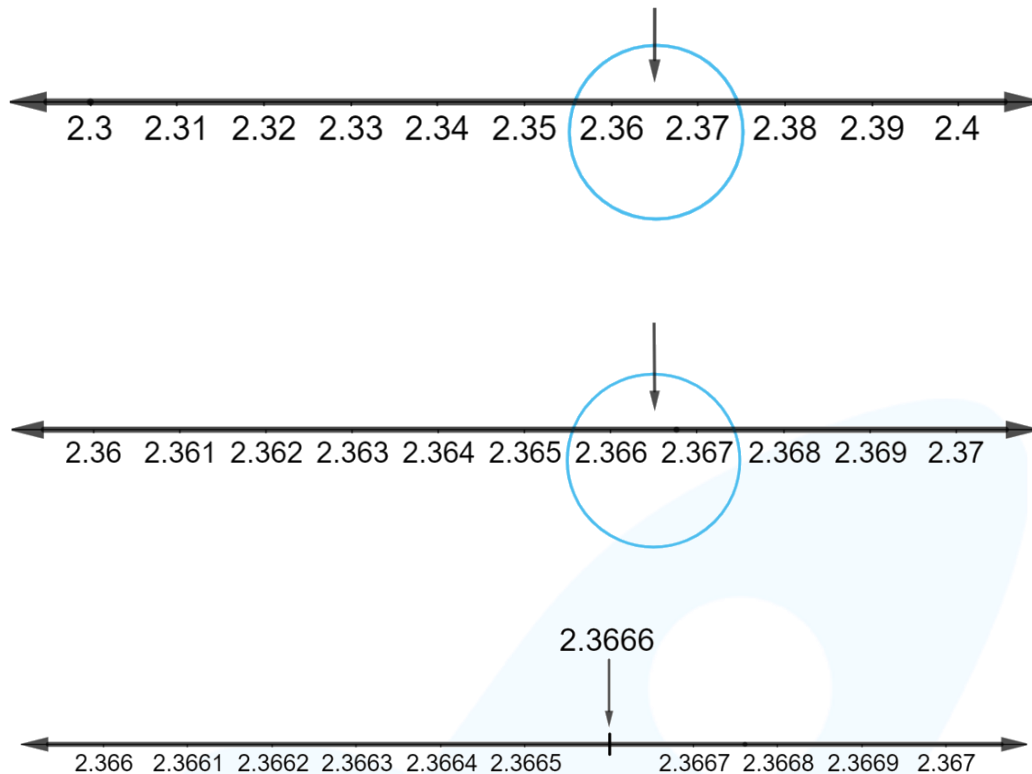


Example: Visualize the representation of $2.3\bar{6}$ on the number line up to 4 decimal places.

Solution: We know that $2.3\bar{6}$ can be written up to 5 decimal places as 2.3666 and lies between whole numbers 2 and 3.

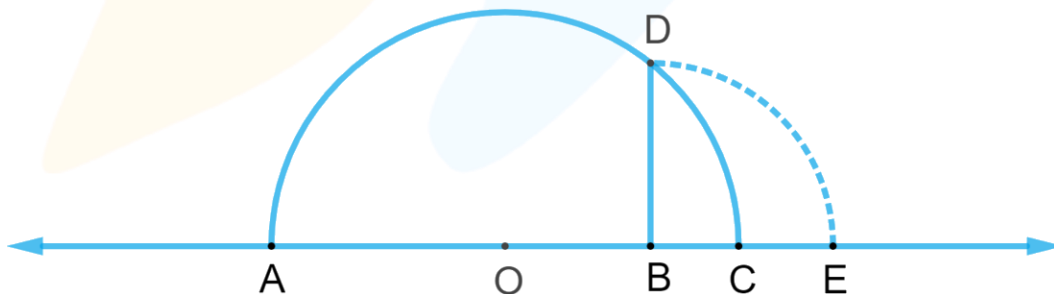
Let us show 2.3666 on the number line through the process of successive magnification.





Example: Represent $\sqrt{4.3}$ on the number line.

Solution: Mark the distance 4.3 units from a fixed-point A on a given line to obtain a point B such that $AB = 3.5$ units. From B, mark a distance of 1 unit and mark the new point as C. Find the midpoint of AC and mark that point as O. Draw a semicircle with center O and radius OC. Draw a line perpendicular to AC passing through B and intersecting the semicircle at D. Then, $BD = \sqrt{4.3}$



Hence the point E shows $\sqrt{4.3}$ on the number line, if B is the origin i.e., 0.

NOTE: Every real number is represented by a unique point on the number line and every point on the number line represents one and only one real number.

Operations on Real Numbers:

As we know that rational numbers satisfy the commutative, associative and distributive laws for addition and multiplication that is rational numbers are 'closed' with respect to addition, subtraction, multiplication and division (except by zero).

Irrational numbers also satisfy the commutative, associative and distributive laws for addition and multiplication. However, the sum, difference, quotients and products of irrational numbers are not always irrational.

- The sum or difference of a rational number and an irrational number is irrational.
- The product or quotient of a non-zero rational number with an irrational number is irrational.
- If we add, subtract, multiply or divide two irrationals, the result may be rational or irrational.

For positive real numbers a and b , the following identities hold:

$$\begin{array}{lll}
 \text{(i) } \sqrt{ab} = \sqrt{a} \times \sqrt{b} & \text{(ii) } \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} & \text{(iii) } (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b \\
 \text{(iv) } (a + \sqrt{b})(a - \sqrt{b}) = a^2 - b & & \text{(v) } (\sqrt{a} + \sqrt{b})^2 = a + 2\sqrt{ab} + b
 \end{array}$$

NOTE: To rationalize the denominator of $\left(\frac{1}{\sqrt{a}+b}\right)$ we multiply this by $\left(\frac{\sqrt{a}-b}{\sqrt{a}-b}\right)$ where a and b are integers.

Let us consider an example:

Example: Rationalize the denominator of $\frac{1}{\sqrt{5}+\sqrt{2}}$.

Solution: Let us multiply by $\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}}$

$$\begin{aligned}\frac{1}{\sqrt{5}+\sqrt{2}} &= \frac{1}{\sqrt{5}+\sqrt{2}} \times \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}} \\ &= \frac{\sqrt{5}-\sqrt{2}}{(\sqrt{5})^2 - (\sqrt{2})^2} \\ &= \frac{\sqrt{5}-\sqrt{2}}{5-2} \\ &= \frac{\sqrt{5}-\sqrt{2}}{3}\end{aligned}$$

Laws of Exponents for Real Numbers:

Let $a > 0$ be a real number and m and n be rational numbers. Then

(i) $a^m \times a^n = a^{m+n}$

(ii) $(a^m)^n = a^{mn}$

(iii) $\frac{a^m}{a^n} = a^{m-n}$

(iv) $a^m \times b^m = (ab)^m$

POLYNOMIALS

Polynomials:

An expression of more than two algebraic terms, that are added, subtracted, or multiplied called a polynomial. It includes at least one variable and typically includes constants and positive exponents as well.

Algebraic expressions of the form $p(x)$ like $3x$, $(2x^2 + 3x)$, $(2x^3 + 3x^2 + x + 2)$ have only whole numbers as the exponents of the variable, are called **polynomials in one variable**. Expressions like $\frac{1}{x+2}$, $\frac{1}{2y^2 - y + 4}$, $\sqrt{x} - 2$ are not polynomials because exponents are not whole numbers.

In the polynomial $(2x^2 + 3x)$, the expressions $2x^2$ and $3x$ are called the **terms** of the polynomial. Similarly, the polynomial $(2x^3 + 3x^2 + 2)$ has three terms, namely, $2x^3$, $3x^2$ and 2 .

Each term of a polynomial has a **coefficient**. So, in $(2x^2 + 3x)$, the coefficient of x^2 is 2, the coefficient of x is 3.

3, 2, -5, -7, etc. are examples of **constant polynomials**. The constant polynomial 0 is called the **zero polynomial**.

Polynomials having only one term are called monomials ('mono' means 'one').

Polynomials having only two terms are called binomials ('bi' means 'two').

Polynomials having only three terms are called trinomials ('tri' means 'three').

If $p(x)$ is a polynomial in x , the **highest power** of x in $p(x)$ is called the **degree** of the polynomial $p(x)$.

For example:

$3x + 1$ is a polynomial in the variable x of degree 1.

$p^2 - 2p - 1$ is a polynomial in the variable p of degree 2.

$2y^3 + 2y^2 - y + 3$ is a polynomial in the variable y of degree 3.

$5m^5 - 2m^4 + 4m^3 + m - 1$ is a polynomial in the variable m of degree 5.

A polynomial of degree 1 is called a **linear polynomial**.

A polynomial of degree 2 is called a **quadratic polynomial**.

A polynomial of degree 3 is called a **cubic polynomial**.

A polynomial $p(x)$ in one variable x of degree n is an algebraic expression in x of the form $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ where $a_0, a_1, a_2, \dots, a_n$ are constants and $a_n \neq 0$

If $a_0 = a_1 = a_2 = \dots = a_n = 0$, i.e., all the constants are zero, we get the **zero polynomial**, which is denoted by 0.

NOTE: The degree of a **non-zero constant polynomial** is **zero** and the degree of the **zero polynomial** is **not defined**.

Zeroes of a Polynomial:

A real number k is said to be a **zero** or **root** of a polynomial $p(x)$, if $p(k) = 0$.

In other words, we say $p(x) = 0$ is a polynomial equation and 1 is the **root** of the polynomial equation $p(x) = 0$. So, we say 1 is the zero of the polynomial $(x-1)$, or a **root** of the polynomial equation $(x-1) = 0$.

NOTE:

- A non-zero constant polynomial has no zero and every real number is a zero of the zero polynomial.
- A linear polynomial has one and only one zero.
- A zero of a polynomial need not be 0.
- 0 may be a zero of a polynomial.
- A polynomial can have more than one zero.

Remainder Theorem:

If $p(x)$ is any polynomial of degree greater than or equal to 1 and $p(x)$ is divided by the linear polynomial $(x-a)$, and a be any real number then the remainder is $p(a)$.

Factor Theorem:

If $p(x)$ is a polynomial of degree greater than or equal to 1 and a is any real number, then

- (i) $(x-a)$ is a factor of $p(x)$, if $p(a)=0$, and
- (ii) $p(a)=0$, if $(x-a)$ is a factor of $p(x)$.

Factorization of the Polynomials:

We will now use the Factor Theorem to factorize some ax^2+bx+c by **splitting the middle term** as follows:

Let its factors be $(px+q)$ and $(rx+s)$

Then,

$$\begin{aligned} ax^2+bx+c &= (px+q)(rx+s) \\ &= prx^2+(ps+qr)x+qs \end{aligned}$$

Comparing the coefficients of x^2 , we get $a = pr$

Similarly, comparing the coefficients of x , we get $b = ps + qr$

And, on comparing the constant terms, we get $c = qs$

This shows us that b is the sum of two numbers ps and qr , whose product is $(ps) \times (qr) = (pr) \times (qs) = ac$.

Therefore, to factorize ax^2+bx+c , we will write b as the sum of two numbers whose product is ac .

Let us consider an example for better understanding.

Example: Factorise $6x^2 + 17x + 5$ by splitting the middle term.

Solution: If we can find two numbers p and q such that $p + q = 17$ and $pq = 6 \times 5 = 30$, then we can get the factors.

So, let us look for the pairs of factors of 30. Some are 1 and 30, 2 and 15, 3 and 10, 5 and 6. Of these pairs, 2 and 15 will give us $p + q = 17$.

So,

$$\begin{aligned} 6x^2 + 17x + 5 &= 6x^2 + (2 + 15)x + 5 \\ &= 6x^2 + 2x + 15x + 5 \\ &= 2x(3x + 1) + 5(3x + 1) \\ &= (3x + 1)(2x + 5) \end{aligned}$$

Therefore, $(3x + 1)$ and $(2x + 5)$ are the factors of the polynomial $6x^2 + 17x + 5$.

Algebraic Identities:

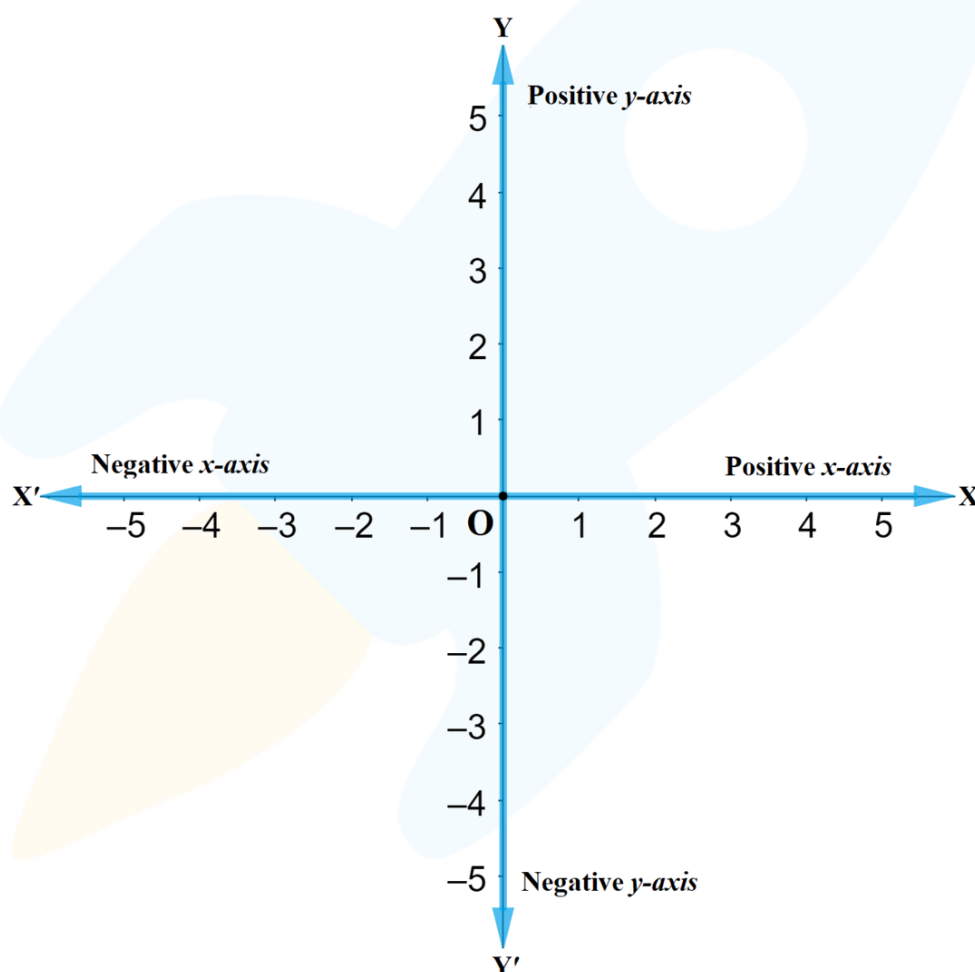
- 1) $(x + y)^2 = x^2 + 2xy + y^2$
- 2) $(x - y)^2 = x^2 - 2xy + y^2$
- 3) $x^2 - y^2 = (x + y)(x - y)$
- 4) $(x + a)(x + b) = x^2 + (a + b)x + ab$
- 5) $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$
- 6) $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$
- 7) $(x - y)^3 = x^3 - y^3 - 3xy(x - y)$
- 8) $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

COORDINATE GEOMETRY

In honor of the great French mathematician Rene Descartes, the system used for describing the position of a point in a plane is known as the Cartesian system.

Cartesian System:

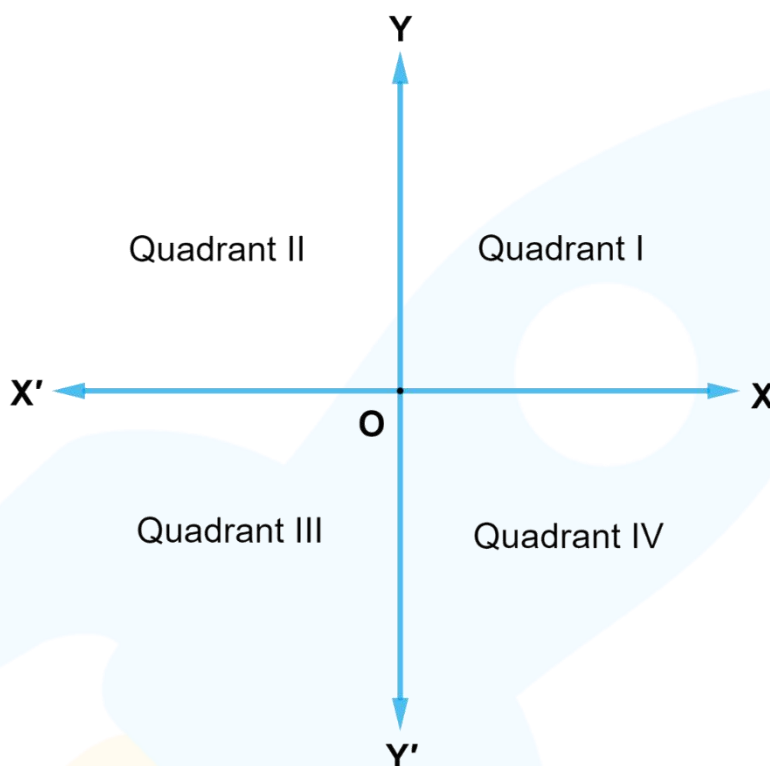
Position of any object lying in a plane can be represented with the help of two perpendicular lines, one horizontal-line and the other vertical-line, in such a way that the two lines cross each other at their zeroes or origin.



The horizontal line XX' is called the **x -axis** and the vertical line YY' is called the **y -axis**. The point of intersection of x -axis and y -axis is called the **origin** and is denoted by O . Since the positive numbers lie on the directions OX and OY , OX and OY are called the **positive directions** of the x -axis and the y -axis, respectively. Similarly, OX' and OY' are called the **negative directions** of the x -axis and the y -axis, respectively.

Cartesian Plane:

We observe that the axes divide the plane into four parts. These four parts are called the **quadrants**, numbered I, II, III and IV anticlockwise from OX. So, the plane consists of the axes and these quadrants. We call the plane, the **Cartesian plane**, or the **coordinate plane**, or the **xy-plane**. The axes are called the **coordinate axes**.



We write the coordinates of a point using the following conventions:

- (i) The ***x*-coordinate** of a point is its perpendicular distance from the *y*-axis measured along the *x*-axis, positive along the positive direction of the *x*-axis and negative along the negative direction of the *x*-axis. The *x*-coordinate is also called the **abscissa**.
- (ii) The ***y*-coordinate** of a point is its perpendicular distance from the *x*-axis measured along the *y*-axis, positive along the positive direction of the *y*-axis and negative along the negative direction of the *y*-axis. The *y*-coordinate is also called the **ordinate**.
- (iii) In stating the coordinates of a point in the coordinate plane, the *x*-coordinate comes first, and then the *y*-coordinate. If the abscissa of a point is x and the ordinate is y , then (x, y) are called the **coordinates** of the point. We place the coordinates in brackets which is called the **ordered pair**.

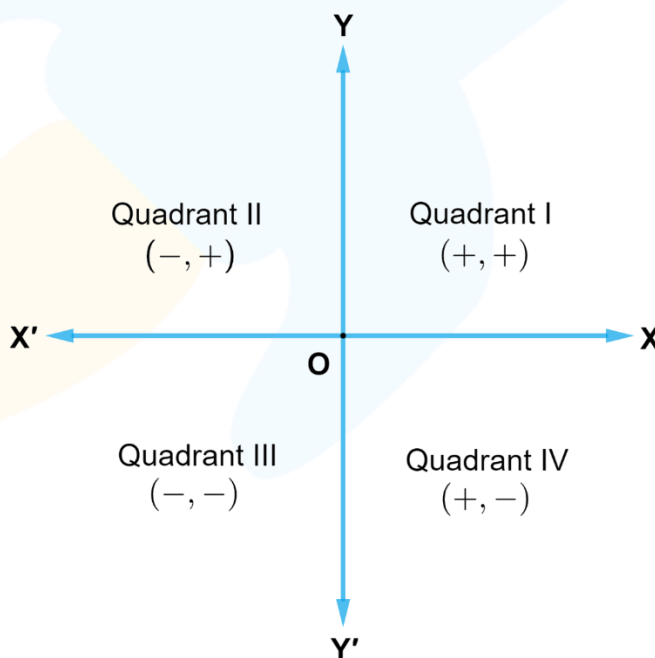
NOTE:

- (i) The coordinates describe a point in the plane uniquely. if $x \neq y$, then the position of (x, y) in the Cartesian plane is different from the position of (y, x) .
- (ii) The coordinates of the origin are $(0, 0)$, since the origin has zero distance from both the axes so that its abscissa and ordinate are both zero.
- (iii) The coordinates of a point on the x -axis are of the form $(x, 0)$ and that of the point on the y -axis are $(0, y)$

Sign of the Coordinates of a Point in a Quadrant:

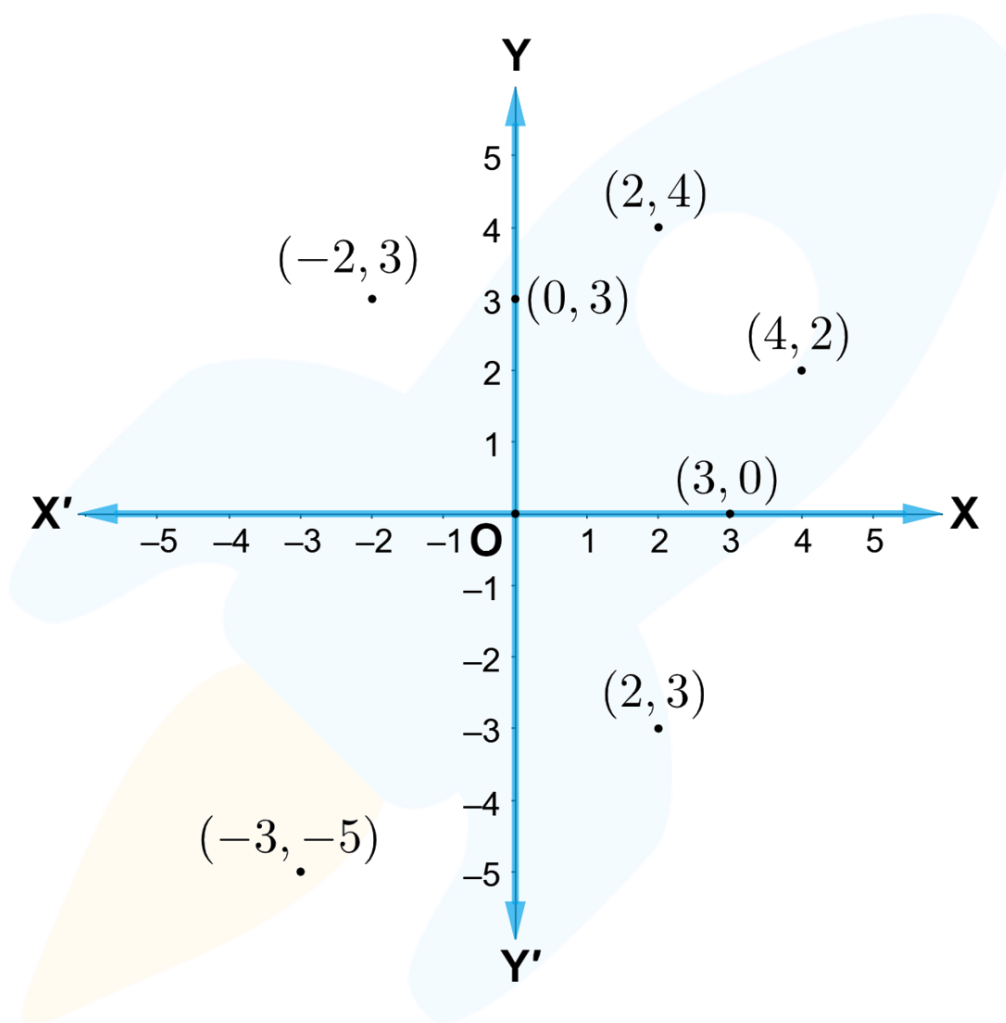
We have observed the following relationship between the signs of the coordinates of a point and the quadrant of a point in which it lies.

- (i) If a point is in the 1st quadrant, then the point will be in the form $(+, +)$, since the 1st quadrant is enclosed by the positive x -axis and the positive y -axis.
- (ii) If a point is in the 2nd quadrant, then the point will be in the form $(-, +)$, since the 2nd quadrant is enclosed by the negative x -axis and the positive y -axis.
- (iii) If a point is in the 3rd quadrant, then the point will be in the form $(-, -)$, since the 3rd quadrant is enclosed by the negative x -axis and the negative y -axis.
- (iv) If a point is in the 4th quadrant, then the point will be in the form $(+, -)$, since the 4th quadrant is enclosed by the positive x -axis and the negative y -axis.



Plotting the Points in a Cartesian Plane:

Let us locate the points $(3, 0)$, $(0, 3)$, $(2, 4)$, $(4, 2)$, $(-2, 3)$, $(2, -3)$ and $(-5, -3)$ in the Cartesian plane.



LINEAR EQUATIONS IN TWO VARIABLES

Any equation of the form $ax + by + c = 0$, where a , b and c are real numbers and a and b both are non-zero, is called a linear equation in two variables x and y .

a is the coefficient of x , b is the coefficient of y and c is the constant term.

Remember:

- (i) if we put $a = 0$, then the equation $ax + by + c = 0$ becomes $by + c = 0$ which is a linear equation in one variable y and no longer a linear equation in two variables.
- (ii) if we put $b = 0$, then the equation $ax + by + c = 0$ becomes $ax + c = 0$ which is a linear equation in one variable x and no longer a linear equation in two variables.
- (iii) if we put $a = 0$, and $b = 0$, then the equation $ax + by + c = 0$ becomes $c = 0$ which is a constant and no longer a linear equation in two variables.

Solution of a linear equation in two variables:

As we can see there are two variables in the equation $ax + by + c = 0$.

So, a solution of the equation means values of each x and y which satisfy the given equation.

Let us take the equation $3x + 4y - 20 = 0$ as an example for better understanding.

Put $x = 0$ and $y = 5$ in the LHS of the equation $3x + 4y - 20 = 0$

$$\begin{aligned} LHS &= 3 \times 0 + 4 \times 5 - 20 \\ &= 0 + 20 - 20 \\ &= 0 = RHS \end{aligned}$$

So, $x = 0$ and $y = 5$ satisfy the equation $3x + 4y - 20 = 0$

We write the solution in ordered pair, to write the solution in ordered pair first we write the value for x and then the value for y

i.e. $(0, 5)$

Therefore, we can see that $(0, 5)$ is a solution for the equation $3x + 4y - 20 = 0$.

Can we find another solution for the same equation?

Let us try to find:

Put $x = 4$ and $y = 2$ in the LHS of the equation $3x + 4y - 20 = 0$

$$\begin{aligned} LHS &= 3 \times 4 + 4 \times 2 - 20 \\ &= 12 + 8 - 20 \\ &= 20 - 20 \\ &= 0 = RHS \end{aligned}$$

So, $x = 4$ and $y = 2$ satisfy the equation $3x + 4y - 20 = 0$

Therefore, $(4, 2)$ is another solution for the equation $3x + 4y - 20 = 0$.

Can we find more?

Let us try one more time:

Put $x = 8$ and $y = -1$ in the LHS of the equation $3x + 4y - 20 = 0$

$$\begin{aligned} LHS &= 3 \times 8 + 4 \times (-1) - 20 \\ &= 24 - 4 - 20 \\ &= 24 - 24 \\ &= 0 = RHS \end{aligned}$$

So, $x = 8$ and $y = -1$ satisfy the equation $3x + 4y - 20 = 0$

Therefore, $(8, -1)$ is another one solution for the equation $3x + 4y - 20 = 0$.

Hence, we can clearly see that $(0, 5)$, $(4, 2)$, $(8, -1)$ are the solutions for the equation $3x + 4y - 20 = 0$.

We will get more solutions for the equation $3x + 4y - 20 = 0$, if we keep trying. So, there is no end to different solutions of a linear equation in two variables.

Conclusion: A linear equation in two variables has infinitely many solutions.

Note: An easy way of getting a solution is to take $x = 0$ and get the corresponding value of y . Similarly, we can put $y = 0$ and obtain the corresponding value of x .

Let us find five different solutions for the equation $2x - 3y = 5$

Simplify the equation $2x - 3y = 5$ to x

$$2x - 3y = 5$$

$$2x = 5 + 3y$$

$$x = \frac{5 + 3y}{2} \quad (i)$$

Now put $y = 0$ in equation (i) to get the value of x

$$x = \frac{5 + 3 \times 0}{2}$$

$$= \frac{5 + 0}{2}$$

$$= \frac{5}{2}$$

Therefore, $\left(\frac{5}{2}, 0\right)$ is a solution for the equation $2x - 3y = 5$.

Now put $y = 1$ in equation (i) to get the value of x

$$x = \frac{5 + 3 \times 1}{2}$$

$$= \frac{5 + 3}{2}$$

$$= \frac{8}{2}$$

$$= 4$$

Therefore, $(4, 1)$ is a solution for the equation $2x - 3y = 5$.

Now put $y = 3$ in equation (i) to get the value of x

$$\begin{aligned}x &= \frac{5+3 \times 3}{2} \\&= \frac{5+9}{2} \\&= \frac{14}{2} \\&= 7\end{aligned}$$

Therefore, $(7, 3)$ is a solution for the equation $2x - 3y = 5$.

Now put $y = -1$ in equation (i) to get the value of x

$$\begin{aligned}x &= \frac{5+3 \times (-1)}{2} \\&= \frac{5-3}{2} \\&= \frac{2}{2} \\&= 1\end{aligned}$$

Therefore, $(1, -1)$ is a solution for the equation $2x - 3y = 5$.

Now put $y = -3$ in equation (i) to get the value of x

$$\begin{aligned}x &= \frac{5+3 \times (-3)}{2} \\&= \frac{5-9}{2} \\&= \frac{-4}{2} \\&= -2\end{aligned}$$

Therefore, $(-2, -3)$ is a solution for the equation $2x - 3y = 5$.

Hence, $\left(\frac{5}{2}, 0\right)$, $(4, 1)$, $(7, 3)$, $(1, -1)$, and $(-2, -3)$ are five different solutions for the equation $2x - 3y = 5$.

Graph of a linear equation in two variables:

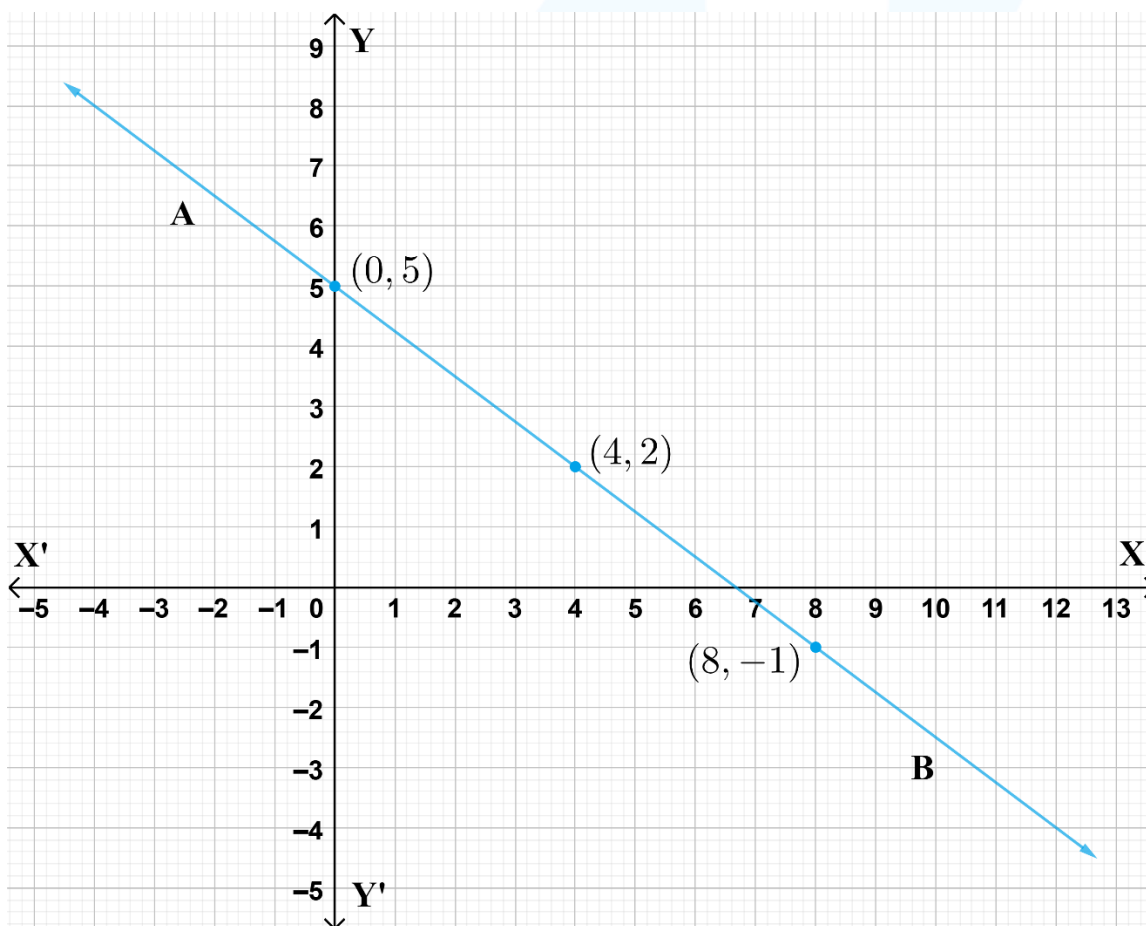
We have learnt to obtain the solutions of a linear equation in two variables algebraically.

Now let us find how to represent the equation geometrically.

The solutions of the linear equation $3x + 4y - 20 = 0$ are $(0, 5)$, $(4, 2)$, $(8, -1)$ and can be expressed in the form of a table as follows by writing the values of x below the corresponding values of y :

x	0	4	8	...
y	5	2	-1	...

Let us plot the points $(0, 5)$, $(4, 2)$, and $(8, -1)$ on a graph paper.



Now we can join any two of these points to obtain a line. Let us say this line AB.

We can clearly see that the other point $(4, 2)$ also lie on the line AB.

Now, let us pick another point on this line AB, say $(12, -4)$ which is also solution for the equation $3x + 4y - 20 = 0$.

Put $x = 12$ and $y = -4$ in the LHS of the equation $3x + 4y - 20 = 0$

$$\begin{aligned} LHS &= 3 \times 12 + 4 \times (-4) - 20 \\ &= 36 - 16 - 20 \\ &= 36 - 36 \\ &= 0 = RHS \end{aligned}$$

So, $x = 12$ and $y = -4$ satisfy the equation $3x + 4y - 20 = 0$

Therefore, $(12, -4)$ is another solution for the equation $3x + 4y - 20 = 0$.

Let us take another point $(-4, 8)$ on the line AB

Put $x = -4$ and $y = 8$ in the LHS of the equation $3x + 4y - 20 = 0$

$$\begin{aligned} LHS &= 3 \times (-4) + 4 \times 8 - 20 \\ &= -12 + 32 - 20 \\ &= 32 - 32 \\ &= 0 = RHS \end{aligned}$$

So, $x = -4$ and $y = 8$ satisfy the equation $3x + 4y - 20 = 0$

Therefore, $(-4, 8)$ is another solution for the equation $3x + 4y - 20 = 0$.

We can pick any other point on this line AB and verify whether its coordinates satisfy the equation or not.

Now, take any point not lying on the line AB, say $(4, 3)$ and check whether the coordinates satisfy the equation:

Put $x = 4$ and $y = 3$ in the LHS of the equation $3x + 4y - 20 = 0$

$$\begin{aligned}LHS &= 3 \times 4 + 4 \times 3 - 20 \\&= 12 + 12 - 20 \\&= 24 - 20 \\&= 4 \neq RHS\end{aligned}$$

So, $x = 4$ and $y = 3$ do not satisfy the equation $3x + 4y - 20 = 0$

Therefore, $(4, 3)$ is not a solution for the equation $3x + 4y - 20 = 0$.

Conclusion:

- (1) Every point whose coordinates satisfy equation $3x + 4y - 20 = 0$ lies on the line AB.
- (2) Every point (a, b) on the line AB gives a solution $x = a, y = b$ of equation $3x + 4y - 20 = 0$
- (3) Any point, which doesn't lie on the line AB, is not a solution of equation $3x + 4y - 20 = 0$

So, we can conclude that every point on the line satisfies the equation of the line and every solution of the equation is a point on the line.

Hence, we can say that linear equation in two variables is represented geometrically by a line whose points make up the collection of solutions of the equation.

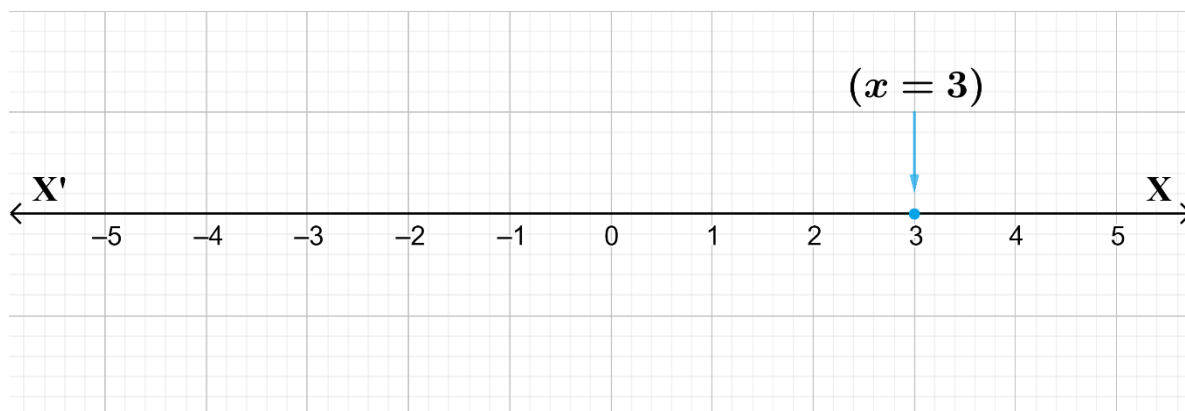
So, this is called the graph of the linear equation and a degree one, polynomial equation $ax + by + c = 0$ is called a linear equation because its geometrical representation is a straight line.

Note: To obtain the graph of a linear equation in two variables, it is enough to plot two points corresponding to two solutions and join them by a line. However, it is advisable to plot more than two such points so that we can immediately check the correctness of the graph.

Equations of lines parallel to the x-axis and y-axis:

Let us consider the equation $x - 3 = 0$ which is a linear equation in one variable only and its unique solution is $x = 3$.

We can show the equation $x - 3 = 0$ as a point on the number line as given below:



Can we rewrite the equation $x - 3 = 0$ in two variables?

Let's rewrite it:

$$\Rightarrow x + 0 \times y - 3 = 0$$

$$\Rightarrow x + 0y - 3 = 0$$

Now, equation $x + 0y - 3 = 0$ has infinitely many solutions and we can write them in the form $(3, b)$ where b is any real number.

Let us check whether that every point of the form $(3, b)$ is a solution of the equation $x + 0y - 3 = 0$.

That is $(3, 1)$ $(3, 4)$ $(3, -2)$ $(3, -3)$ and many more

Let us take $(3, 1)$

Put $x = 3$ and $y = 1$ in the LHS of the equation $x + 0y - 3 = 0$

$$LHS = 3 + 0 \times 1 - 3$$

$$= 3 + 0 - 3$$

$$= 0 = RHS$$

So, $x = 3$ and $y = 1$ satisfy the equation $x + 0y - 3 = 0$

Therefore, $(3, 1)$ is a solution for the equation $x + 0y - 3 = 0$.

Now, Put $x = 3$ and $y = 4$ in the LHS of the equation $x + 0y - 3 = 0$

$$LHS = 3 + 0 \times 4 - 3$$

$$= 3 + 0 - 3$$

$$= 0 = RHS$$

So, $x = 3$ and $y = 4$ satisfy the equation $x + 0y - 3 = 0$

Therefore, $(3, 4)$ is a solution for the equation $x + 0y - 3 = 0$.

Now, Put $x = 3$ and $y = -2$ in the LHS of the equation $x + 0y - 3 = 0$

$$\begin{aligned} LHS &= 3 + 0 \times (-2) - 3 \\ &= 3 - 0 - 3 \\ &= 0 = RHS \end{aligned}$$

So, $x = 3$ and $y = -2$ satisfy the equation $x + 0y - 3 = 0$

Therefore, $(3, -2)$ is a solution for the equation $x + 0y - 3 = 0$.

Now, Put $x = 3$ and $y = -3$ in the LHS of the equation $x + 0y - 3 = 0$

$$\begin{aligned} LHS &= 3 + 0 \times (-3) - 3 \\ &= 3 - 0 - 3 \\ &= 0 = RHS \end{aligned}$$

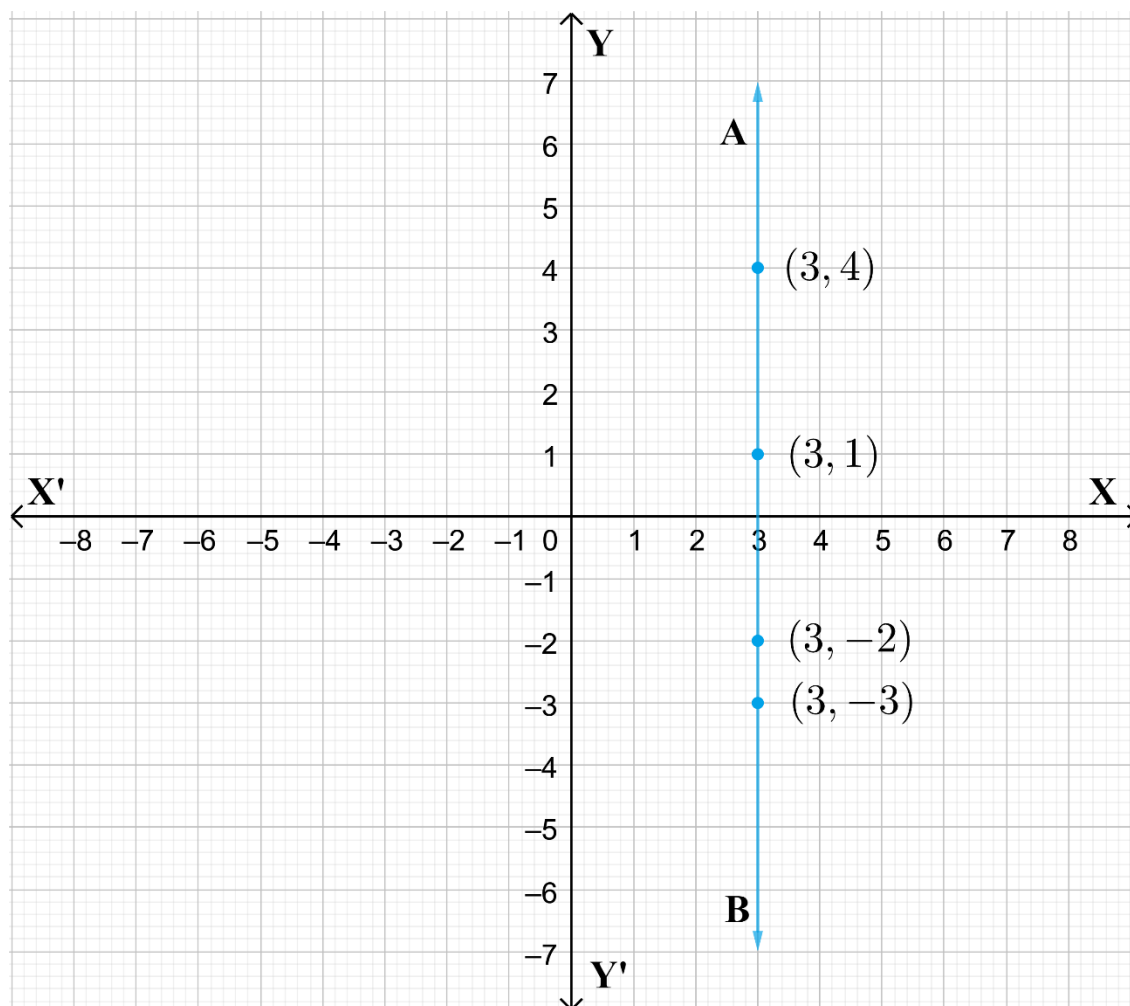
So, $x = 3$ and $y = -3$ satisfy the equation $x + 0y - 3 = 0$

Therefore, $(3, -3)$ is a solution for the equation $x + 0y - 3 = 0$.

The solutions of the linear equation $x + 0y - 3 = 0$ are $(3, 1)$, $(3, 4)$, $(3, -2)$, $(3, -3)$ and can be expressed in the form of a table as follows by writing the values of x below the corresponding values of y :

x	3	3	3	3	...
y	1	4	-2	-3	...

Let us plot the points $(3, 1)$, $(3, 4)$, $(3, -2)$ and $(3, -3)$ on a graph paper.

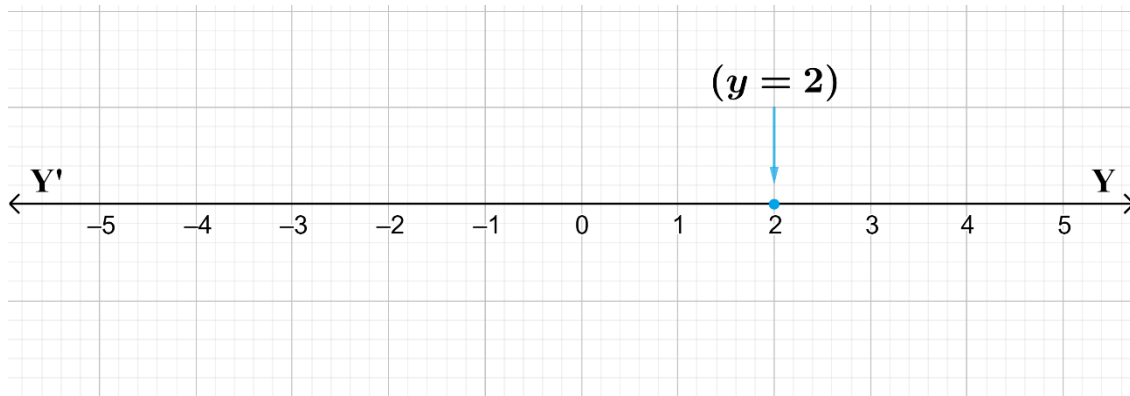


So as, an equation in two variables, $x - 3 = 0$ is represented by the line AB in the graph which is parallel to the y-axis.

Conclusion: Any equation $ax + 0y + c = 0$ can be represented geometrically in the graph by a line parallel to y-axis.

Let us consider another equation $y - 2 = 0$ which is a linear equation in one variable only and its unique solution is $y = 2$.

We can show the equation $y - 2 = 0$ as a point on the number line as given below:



Can we rewrite the equation $y - 2 = 0$ in two variables?

Let's rewrite it:

$$\Rightarrow 0 \times x + y - 2 = 0$$

$$\Rightarrow 0x + y - 2 = 0$$

Now, equation $0x + y - 2 = 0$ has infinitely many solutions and we can write them in the form $(a, 2)$ where a is any real number.

Let us check whether that every point of the form $(a, 2)$ is a solution of the equation $0x + y - 2 = 0$.

That is $(3, 2)$ $(2, 2)$ $(-1, 2)$ $(-3, 2)$ and many more

Let us take $(3, 2)$

Put $x = 3$ and $y = 2$ in the LHS of the equation $0x + y - 2 = 0$

$$\begin{aligned} LHS &= 0 \times 3 + 2 - 2 \\ &= 0 + 2 - 2 \\ &= 0 = RHS \end{aligned}$$

So, $x = 3$ and $y = 2$ satisfy the equation $0x + y - 2 = 0$

Therefore, $(3, 2)$ is a solution for the equation $0x + y - 2 = 0$.

Now, Put $x = 2$ and $y = 2$ in the LHS of the equation $0x + y - 2 = 0$

$$\begin{aligned} LHS &= 0 \times 2 + 2 - 2 \\ &= 0 + 2 - 2 \\ &= 0 = RHS \end{aligned}$$

So, $x = 2$ and $y = 2$ satisfy the equation $0x + y - 2 = 0$

Therefore, (2, 2) is a solution for the equation $0x + y - 2 = 0$.

Now, Put $x = -1$ and $y = 2$ in the LHS of the equation $0x + y - 2 = 0$

$$\begin{aligned} LHS &= 0 \times (-1) + 2 - 2 \\ &= 0 + 2 - 2 \\ &= 0 = RHS \end{aligned}$$

So, $x = -1$ and $y = 2$ satisfy the equation $0x + y - 2 = 0$

Therefore, (-1, 2) is a solution for the equation $0x + y - 2 = 0$.

Now, Put $x = -3$ and $y = 2$ in the LHS of the equation $0x + y - 2 = 0$

$$\begin{aligned} LHS &= 0 \times (-3) + 2 - 2 \\ &= 0 + 2 - 2 \\ &= 0 = RHS \end{aligned}$$

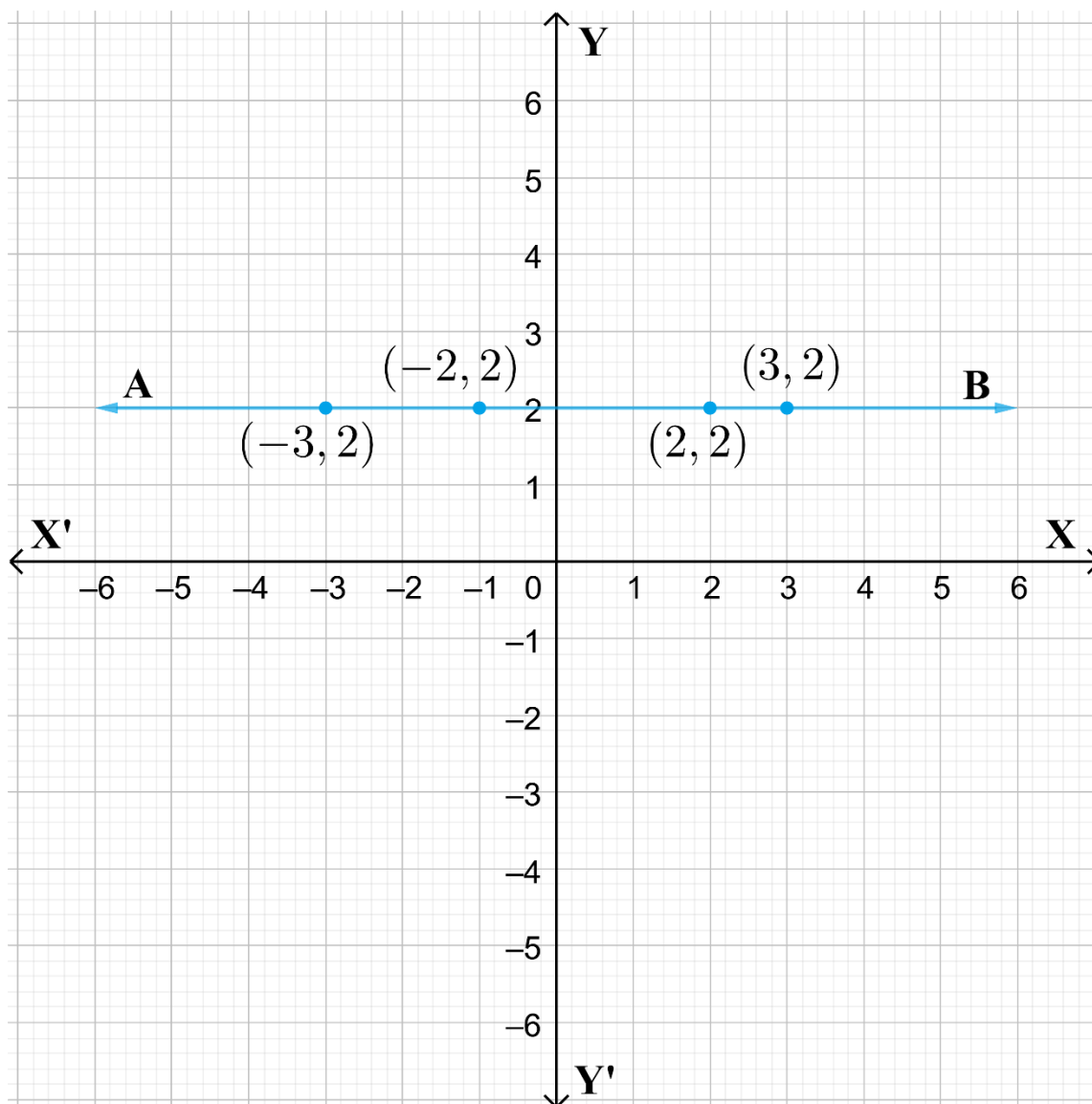
So, $x = -3$ and $y = 2$ satisfy the equation $0x + y - 2 = 0$

Therefore, (-3, 2) is a solution for the equation $0x + y - 2 = 0$.

The solutions of the linear equation $0x + y - 2 = 0$ are (3, 2), (2, 2), (-1, 2), (-3, 2) and can be expressed in the form of a table as follows by writing the values of x below the corresponding values of y :

x	3	2	-1	-3	...
y	2	2	2	2	...

Let us plot the points (3, 2), (2, 2), (-1, 2) and (-3, 2) on a graph paper.



So as, an equation in two variables, $y - 2 = 0$ is represented by the line AB in the graph which is parallel to the x-axis.

Conclusion: Any equation $0x + by + c = 0$ can be represented geometrically in the graph by a line parallel to x-axis.

NOTE:

- $x = 0$ is the equation of the y-axis and $y = 0$ is the equation of the x-axis.
- The graph of $x = a$ is a straight line parallel to the y-axis.
- The graph of $y = a$ is a straight line parallel to the x-axis.

INTRODUCTION TO EUCLID'S GEOMETRY

Euclid, a teacher of mathematics at Alexandria in Egypt, collected all the known work and arranged it in his famous treatise, called 'Elements'. He divided the 'Elements' into thirteen chapters, each called a book. These books influenced the whole world's understanding of geometry for generations to come.

Consider the three steps from solids to points (solids-surfaces-lines-points). In each step we lose one extension, also called a dimension. So, a solid has three dimensions, a surface has two, a line has one and a point has none. Euclid summarized these statements as definitions. He began his exposition by listing 23 definitions in Book 1 of the 'Elements'. A few of them are given below:

1. A point is that which has no part.
2. A line is breadth less length.
3. The ends of a line are points.
4. A straight line is a line which lies evenly with the points on itself.
5. A surface is that which has length and breadth only.
6. The edges of a surface are lines.
7. A plane surface is a surface which lies evenly with the straight lines on itself.

Axioms:

The basic facts which are taken for granted, without proof and which are used throughout in the mathematics are called axioms

Some of **Euclid's axioms**, not in his order, are given below:

- 1) Things which are equal to the same thing are equal to one another.
- 2) If equals are added to equals, the wholes are equal.
- 3) If equals are subtracted from equals, the remainders are equal.
- 4) Things which coincide with one another are equal to one another.
- 5) The whole is greater than the part.
- 6) Things which are double of the same things are equal to one another.
- 7) Things which are halves of the same things are equal to one another.

Axiom 1: Given two distinct points, there is a unique line that passes through them.

Postulates:

The basic facts which are taken, for granted, without proof and which are specific to geometry are called postulates.

Euclid's five postulates:

Postulate 1: A straight line may be drawn from any one point to any other point.

Postulate 2: A terminated line can be produced indefinitely.

Postulate 3: A circle can be drawn with any centre and any radius.

Postulate 4: All right angles are equal to one another.

Postulate 5: If a straight line falling on two straight lines makes the interior angles on the same side of it taken together less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the sum of angles is less than two right angles.

Equivalent Versions of Euclid's Fifth Postulate:

For every line l and for every point P not lying on l , there exists a unique line m passing through P and parallel to l

This result can also be stated in the following form:

Two distinct intersecting lines cannot be parallel to the same line.

Theorem:

The conclusions obtained through logical reasoning based on previously proved results and some axioms constitute a statement known as a theorem or a proposition.

Theorem 1: Two distinct lines cannot have more than one point in common.

LINES AND ANGLES

Line:

A line is a straight path with no thickness and end points, it can be extended infinitely either sides. It is denoted by $\overleftrightarrow{AB}$.



Line-segment:

A line with two endpoints is called a line-segment and it is denoted by $\overline{AB}$.



Ray:

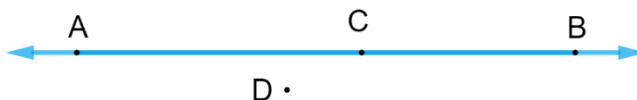
A part of a line with one endpoint is called a ray and it is denoted by $\overrightarrow{AB}$.



Collinear Points:

If three or more points lie on the same line, they are called **collinear** points; otherwise they are called **non-collinear** points.

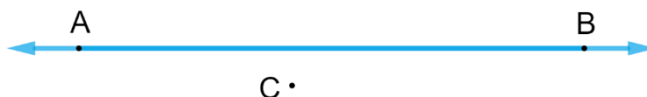
In the figure A, B and C are collinear points.



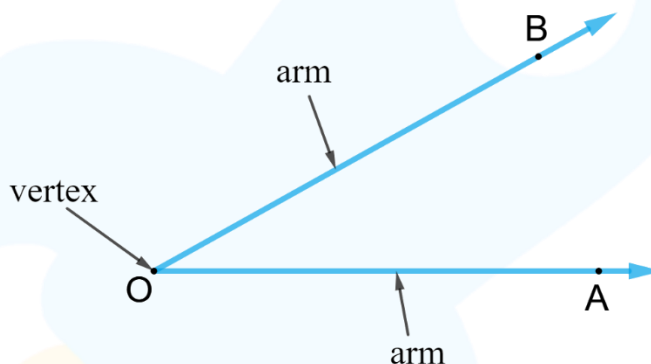
Non-collinear Points:

If three or more points don't lie on the same line, they are called **non-collinear** points.

In the figure A, B and C are non-collinear points.

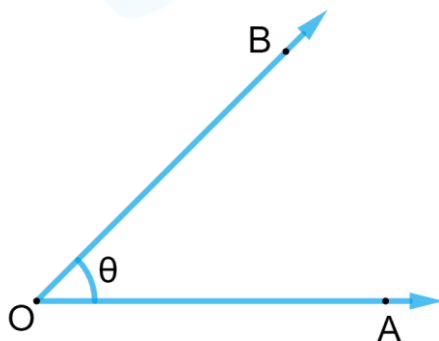
**Angle:**

An angle is formed when two rays originate from the same point. The rays making an angle are called the **arms** of the angle and the common end point is called the **vertex** of the angle.

**1) Acute Angle:**

An angle measured between 0° and 90° is called an acute angle.

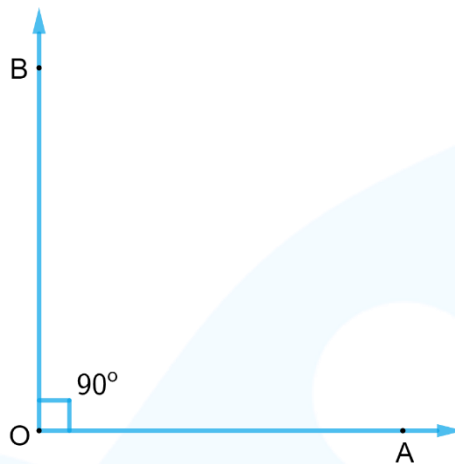
i.e., $0^\circ < \theta < 90^\circ$



2) Right Angle:

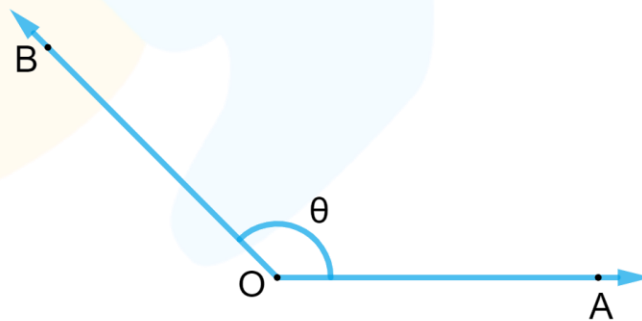
An angle measured exactly equal to 90° is called a right angle.

i.e., $\theta = 90^\circ$

**3) Obtuse Angle:**

An angle which is greater than 90° but less than 180° is called an obtuse angle.

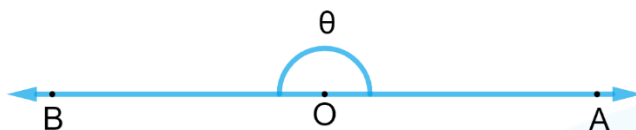
i.e., $90^\circ < \theta < 180^\circ$



4) Straight Angle:

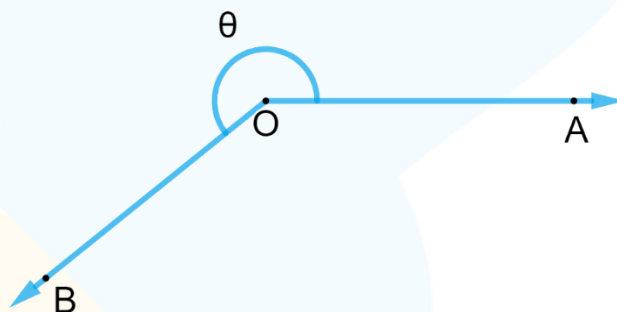
An angle measured exactly equal to 180° is called a straight angle.

i.e., $\theta = 180^\circ$

**5) Reflex Angle:**

An angle which is greater than 180° but less than 360° is called a reflex angle.

i.e., $180^\circ < \theta < 360^\circ$

**Complementary Angles:**

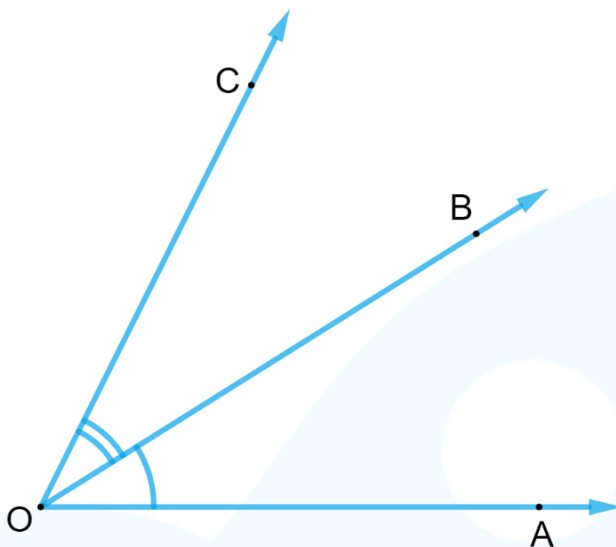
Two angles whose sum is 90° are called complementary angles.

Supplementary Angles:

Two angles whose sum is 180° are called supplementary angles.

Adjacent Angles:

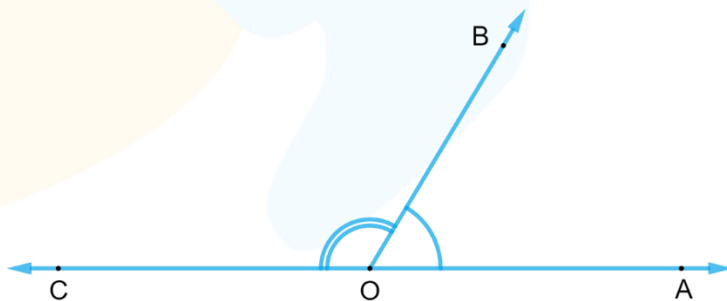
Two angles are said to be adjacent, if they have a common vertex, a common arm and their non-common arms are on different sides of the common arm.



In the figure $\angle AOB$ and $\angle BOC$ are adjacent angles as they have a common vertex O , a common arm OB and their non-common arms OA and OC are on different sides of the common arm OB .

Linear Pair of Angles:

A pair of adjacent angles are called linear pair, if they are supplementary that is their sum is 180° .



In the figure $\angle AOB$ and $\angle BOC$ are adjacent angles and if $\angle AOB + \angle BOC = 180^\circ$, then they are called a linear pair.

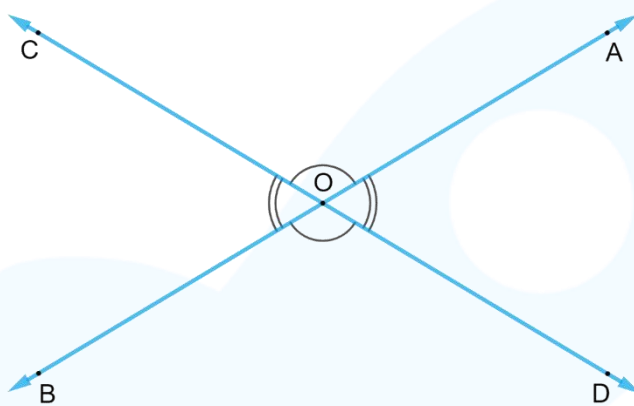
Linear Pair Axiom:

- 1) If a ray stands on a line, then the sum of two adjacent angles so formed is 180° .
- 2) If the sum of two adjacent angles is 180° , then the non-common arms of the angles form a line.

The two axioms above together are called the **Linear Pair Axiom**.

Vertically Opposite Angles:

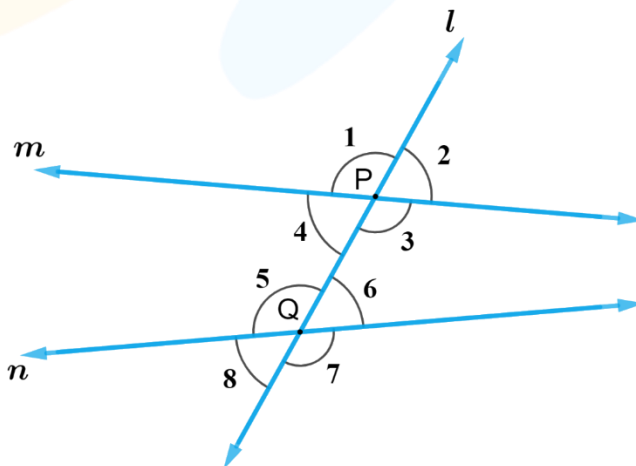
If two lines intersect each other, then the vertically opposite angles are equal.



Lines AB and AC are intersecting at point O, Then vertically opposite angles $\angle AOC = \angle BOD$ and $\angle AOD = \angle BOC$

Parallel Lines and a Transversal:

When a transversal intersects two lines then some pairs of angles formed.



In the figure transversal l intersects lines m and n and pairs of angles formed are as follows:

1) Corresponding angles:

- (a) $\angle 1$ and $\angle 5$
- (b) $\angle 2$ and $\angle 6$
- (c) $\angle 4$ and $\angle 8$
- (d) $\angle 3$ and $\angle 7$

2) Alternate interior angles:

- a) $\angle 4$ and $\angle 6$
- b) $\angle 3$ and $\angle 5$

3) Alternate exterior angles:

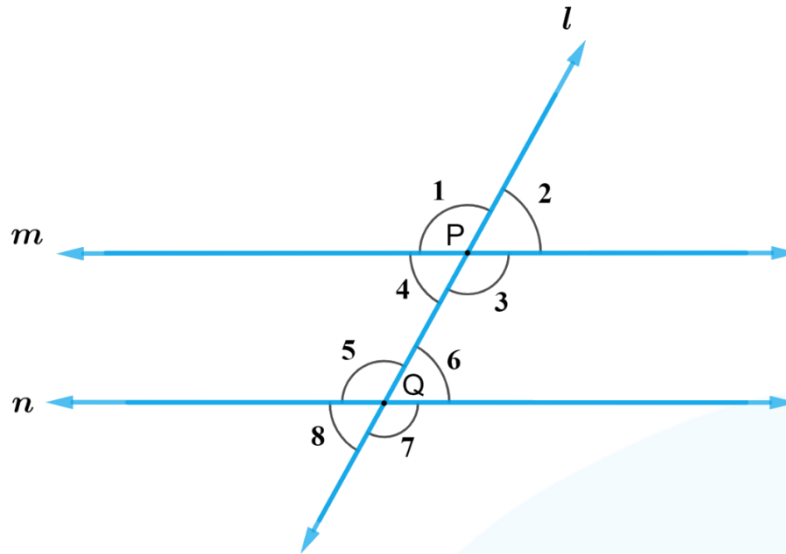
- a) $\angle 1$ and $\angle 7$
- b) $\angle 2$ and $\angle 8$

4) Interior angles on the same side of the transversal:

- a) $\angle 4$ and $\angle 5$
- b) $\angle 3$ and $\angle 6$

If a transversal intersects two parallel lines, then

- (i) each pair of corresponding angles is equal,
- (ii) each pair of alternate interior angles is equal,
- (iii) each pair of interior angles on the same side of the transversal is supplementary



In the figure transversal l intersects parallel lines m and n then,

1) Corresponding angles:

- (a) $\angle 1 = \angle 5$
- (b) $\angle 2 = \angle 6$
- (c) $\angle 4 = \angle 8$
- (d) $\angle 3 = \angle 7$

2) Alternate interior angles:

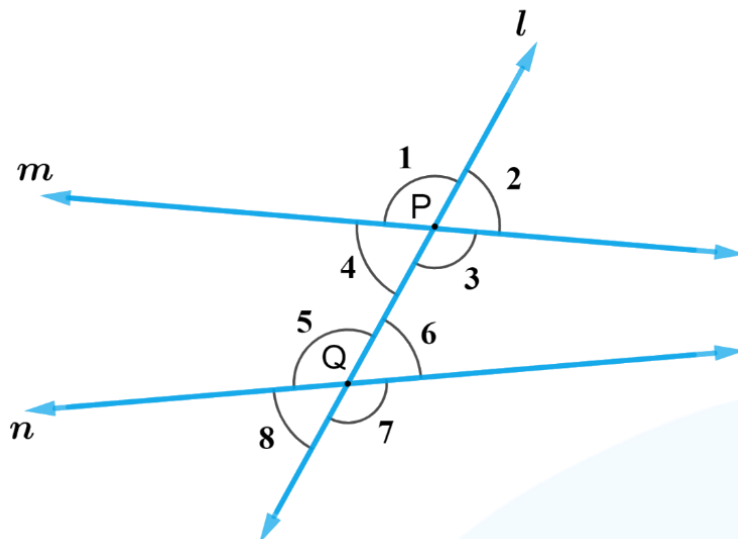
- a) $\angle 4 = \angle 6$
- b) $\angle 3 = \angle 5$

3) Interior angles on the same side of the transversal:

- a) $\angle 4 + \angle 5 = 180^\circ$
- b) $\angle 3 + \angle 6 = 180^\circ$

If a transversal intersects two lines such that, either

- (i) any one pair of corresponding angles is equal, or
 - (ii) any one pair of alternate interior angles is equal, or
 - (iii) any one pair of interior angles on the same side of the transversal is supplementary,
- then the lines are parallel.**



In the figure transversal l intersects lines m and n and if,

1) Corresponding angles:

- (a) $\angle 1 = \angle 5$
- (b) $\angle 2 = \angle 6$
- (c) $\angle 4 = \angle 8$
- (d) $\angle 3 = \angle 7$

or

2) Alternate interior angles:

- c) $\angle 4 = \angle 6$
- a) $\angle 3 = \angle 5$

or

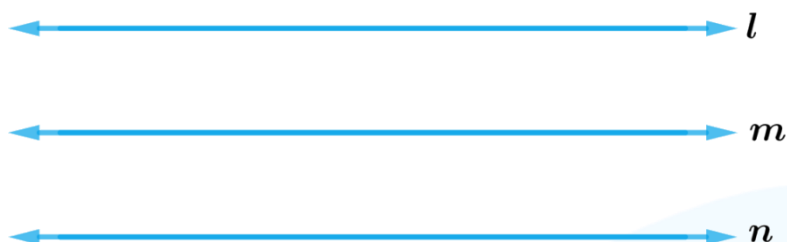
3) Interior angles on the same side of the transversal:

- a) $\angle 4 + \angle 5 = 180^\circ$
- b) $\angle 3 + \angle 6 = 180^\circ$

then the lines are parallel.

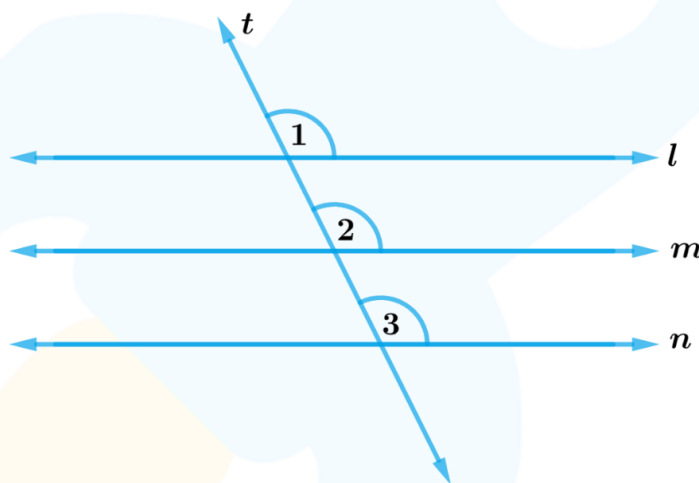
Lines Parallel to the Same Line:

Lines which are parallel to the same line are parallel to each other.



If line $l \parallel$ line m and line $l \parallel$ line n then lines $m \parallel n$.

Let us draw a line t transversal for the lines, l , m and n . and it is given that line $l \parallel$ line m and line $l \parallel$ line n .



Here, line $l \parallel$ line m ,

therefore, $\angle 1 = \angle 2$ (Corresponding angles axiom)

and line $l \parallel$ line n ,

therefore, $\angle 1 = \angle 3$ (Corresponding angles axiom)

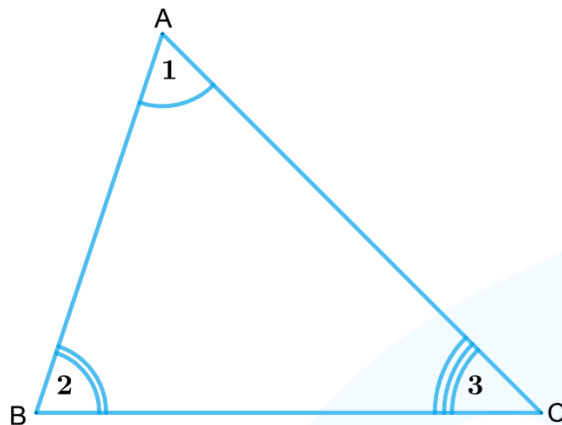
So, $\angle 2 = \angle 3$

But, $\angle 2$ and $\angle 3$ are corresponding angles and they are equal.

Therefore, we can say that lines $m \parallel n$.

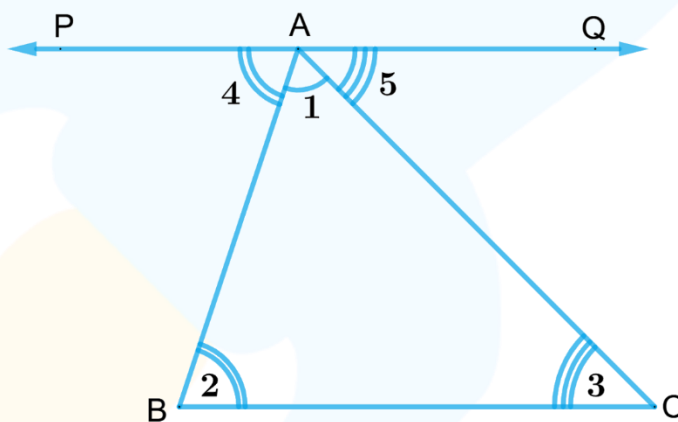
Angle Sum Property of a Triangle:

The sum of the angles of a triangle is 180° .



In $\triangle ABC$, $\angle 1 + \angle 2 + \angle 3 = 180^\circ$

To prove this let us draw a line PAQ parallel to side BC, i.e., $PQ \parallel BC$ through A.



Now, PAQ is a straight line.

Therefore, $\angle 4 + \angle 1 + \angle 5 = 180^\circ$ (i)

But $PQ \parallel BC$ and AB, AC are transversals.

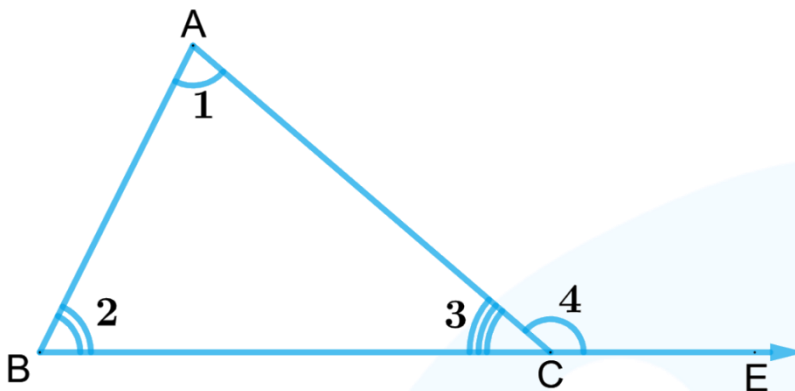
So, $\angle 4 = \angle 2$ and $\angle 5 = \angle 3$ (Pairs of alternate interior angles)

Substituting $\angle 4$ and $\angle 5$ in equation (i), we get $\angle 2 + \angle 1 + \angle 3 = 180^\circ$

That is, $\angle 1 + \angle 2 + \angle 3 = 180^\circ$

Exterior Angle Property of a Triangle:

If a side of a triangle is produced, then the exterior angle so formed is equal to the sum of the two interior opposite angles.



In $\triangle ABC$, BC is produced to BE then $\angle ACE$ is called exterior angle.

Hence, $\angle 4 = \angle 1 + \angle 2$

To prove let us consider $\triangle ABC$ in which

$$\angle 1 + \angle 2 + \angle 3 = 180^\circ \quad (\text{i})$$

Now, BCE is a straight line

$$\text{Therefore, } \angle 3 + \angle 4 = 180^\circ \quad (\text{ii})$$

By using equations (i) and (ii), we get

$$\angle 1 + \angle 2 + \angle 3 = \angle 3 + \angle 4$$

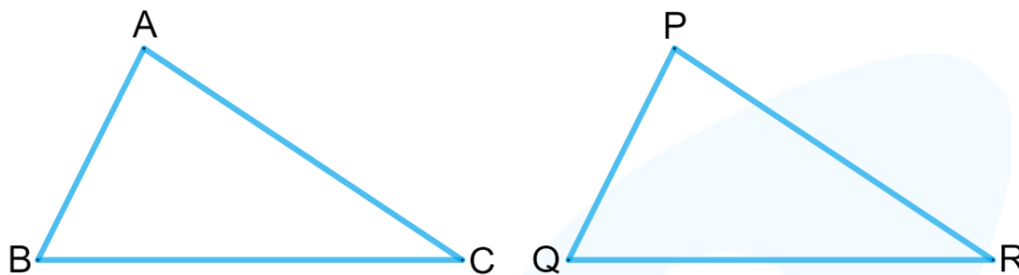
That is, $\angle 4 = \angle 1 + \angle 2$

NOTE: An exterior angle of a triangle is greater than either of its interior opposite angles.

TRIANGLES

Congruence of Triangle:

Two triangles are congruent if the sides and angles of one triangle are equal to the corresponding sides and angles of the other triangle.



If $\triangle ABC$ is congruent to $\triangle PQR$, we write $\triangle ABC \cong \triangle PQR$.

When $\triangle ABC \cong \triangle PQR$, then sides of $\triangle PQR$ fall on corresponding equal sides of $\triangle ABC$ and so is the case for the angles. That is, PQ covers AB, QR covers BC and RP covers CA; $\angle P$ covers $\angle A$, $\angle Q$ covers $\angle B$ and $\angle R$ covers $\angle C$.

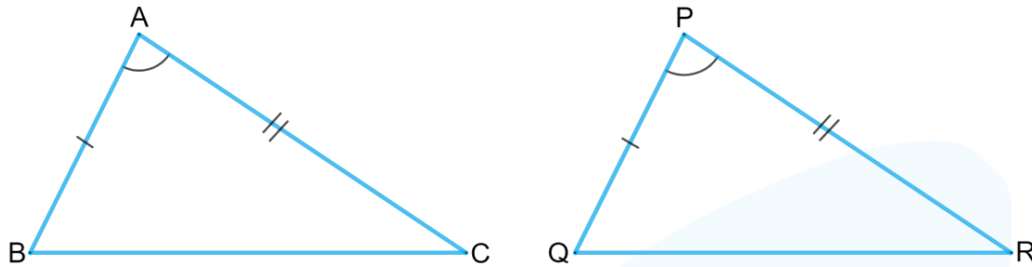
Also, there is a one-one correspondence between the vertices. That is, P corresponds to A, Q to B, and R to C, which is written as $P \leftrightarrow A$, $Q \leftrightarrow B$, $R \leftrightarrow C$.

NOTE:

- 1) It is necessary to write the correspondence of vertices correctly for writing of congruence of triangles in symbolic form i.e., if $P \leftrightarrow A$, $Q \leftrightarrow B$, $R \leftrightarrow C$, under this correspondence, $\triangle ABC \cong \triangle PQR$; but it will not be correct to write $\triangle ABC \cong \triangle QRP$.
- 2) Corresponding parts of congruent triangles are **equal**, and we write in short ‘CPCT’.

Criteria for Congruence of Triangle:

Side-Angle-Side (SAS) congruence rule: Two triangles are congruent if two sides and the included angle of one triangle are equal to the two sides and the included angle of the other triangle.

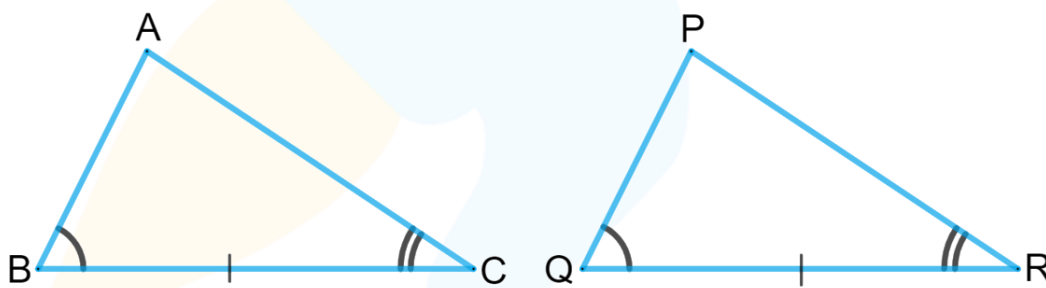


In $\triangle ABC$ and $\triangle PQR$, we have $AB = PQ$, $AC = PR$ and $\angle A = \angle P$

Then, $\triangle ABC \cong \triangle PQR$.

NOTE: SAS congruence rule is accepted true as an **axiom** hence it cannot be proved.

Angle-Side-Angle (ASA) congruence rule: Two triangles are congruent if two angles and the included side of one triangle are equal to two angles and the included side of other triangle.



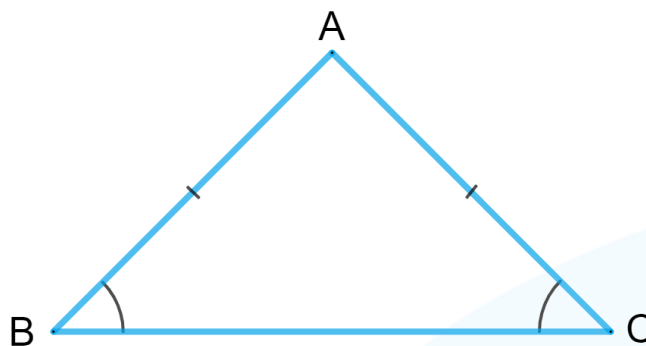
In $\triangle ABC$ and $\triangle PQR$, we have $\angle B = \angle Q$, $\angle C = \angle R$ and $BC = QR$

Then, $\triangle ABC \cong \triangle PQR$.

NOTE: Two triangles are congruent if any two pairs of angles and one pair of corresponding sides are equal. It is called the **Angle-Angle-Side (AAS) Congruence Rule**.

Some Properties of a Triangle:

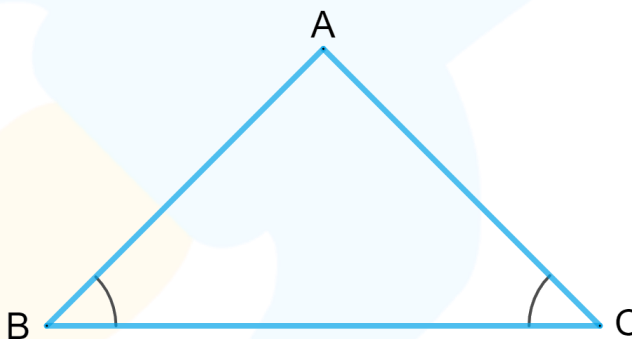
- 1) Angles opposite to equal sides of an isosceles triangle are equal.



In isosceles $\triangle ABC$ we have $AB = AC$. Angles opposite to equal sides AB and AC are $\angle C$ and $\angle B$ respectively.

Then, $\angle B = \angle C$.

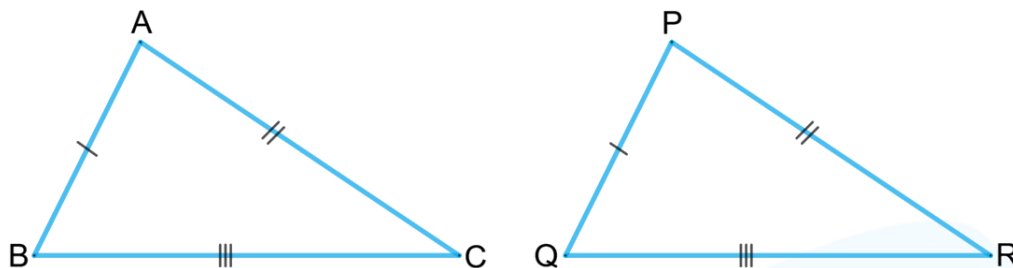
- 2) The sides opposite to equal angles of a triangle are equal.



In $\triangle ABC$ we have $\angle B = \angle C$. Sides opposite to equal angles $\angle C$ and $\angle B$ are AB and AC respectively.

Then, $AB = AC$

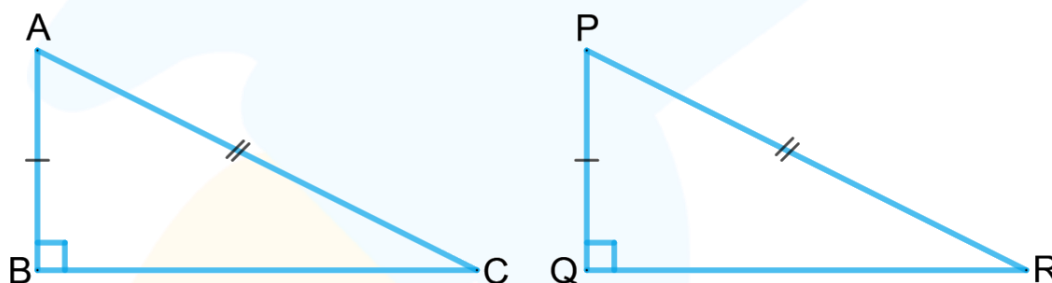
Side-Side-Side (SSS) congruence rule: If three sides of one triangle are equal to the three sides of another triangle, then the two triangles are congruent.



In $\triangle ABC$ and $\triangle PQR$, we have $AB = PQ$, $AC = PR$ and $BC = QR$

Then, $\triangle ABC \cong \triangle PQR$.

Right Angle-Hypotenuse-Side (RHS) congruence rule: If in two right triangles the hypotenuse and one side of one triangle are equal to the hypotenuse and one side of the other triangle, then the two triangles are congruent.

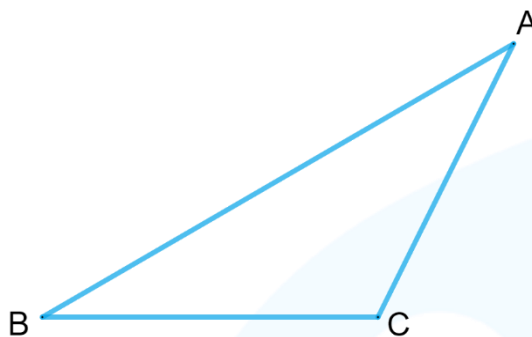


In right triangles $\triangle ABC$ and $\triangle PQR$, we have hypotenuses $AC = PR$ and sides $AB = PQ$.

Then, $\triangle ABC \cong \triangle PQR$.

Inequalities in a Triangle:

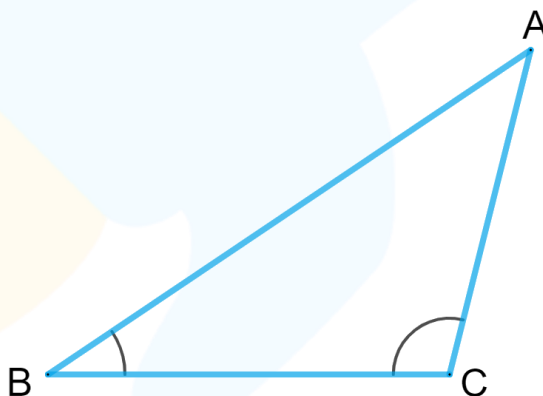
- 1) If two sides of a triangle are unequal, the angle opposite to the longer side is larger (or greater).



In $\triangle ABC$ we have $AB > AC$. Angles opposite to sides AB and AC are $\angle C$ and $\angle B$ respectively.

Then, $\angle C > \angle B$

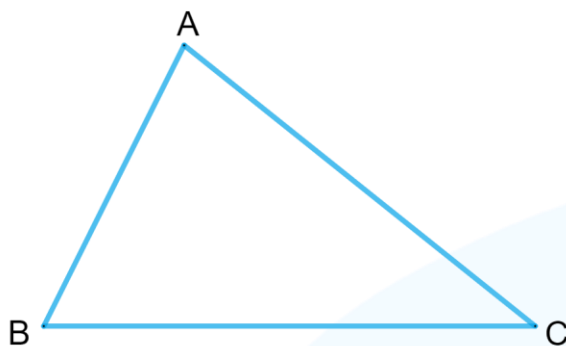
- 2) In any triangle, the side opposite to the larger (greater) angle is longer.



In $\triangle ABC$ we have $\angle C > \angle B$. Sides opposite to angles $\angle C$ and $\angle B$ are AB and AC respectively.

Then, $AB > AC$

3) The sum of any two sides of a triangle is greater than the third side.



In $\triangle ABC$ we have

$$AB + AC > BC$$

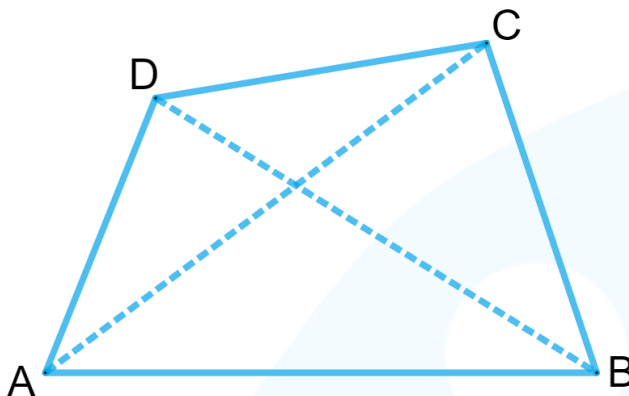
$$AB + BC > AC$$

$$AC + BC > AB$$

QUADRILATERALS

Quadrilaterals:

A figure formed by joining four points in an order is called a quadrilateral. A quadrilateral has **4 sides**, **4 angles** and **4 vertices**.

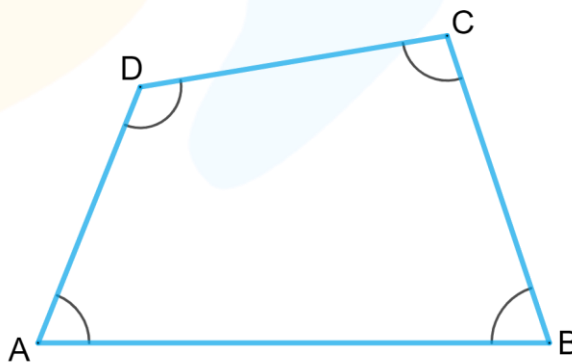


In quadrilateral ABCD, AB, BC, CD and DA are the four sides; A, B, C and D are the four vertices; $\angle A$, $\angle B$, $\angle C$ and $\angle D$ are the four angles formed at the vertices.

Now by joining the opposite vertices A to C and B to D, AC and BD are the **two diagonals** of the quadrilateral ABCD.

Angle Sum Property of a Quadrilaterals:

The sum of the angles of a quadrilateral is 360° .

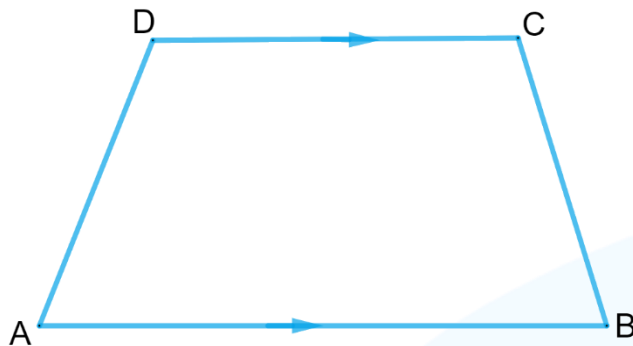


In quadrilateral ABCD

$$\angle A + \angle B + \angle C + \angle D = 360^\circ$$

Types of Quadrilaterals:

If only one pair of opposite sides of quadrilateral are parallel, then it is called a **trapezium**.



In quadrilateral ABCD, only one pair of opposite sides are parallel i.e., $AB \parallel DC$

Therefore, ABCD is a trapezium.

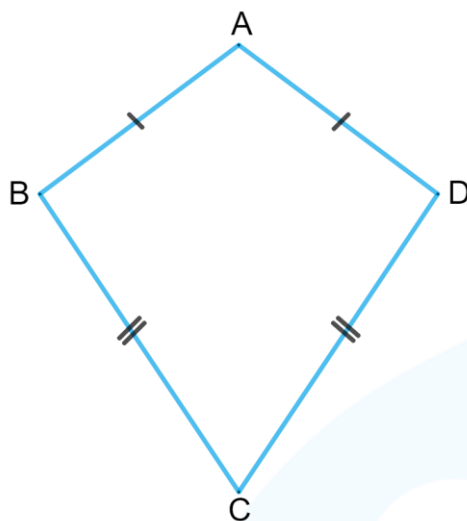
If both pairs of opposite sides of a quadrilateral are parallel, then such a quadrilateral is called a **parallelogram**.



In quadrilateral ABCD, both pairs of opposite sides are parallel i.e., $AB \parallel DC$ and $AD \parallel BC$

Therefore, ABCD is a parallelogram.

If in quadrilateral two pairs of adjacent sides are equal, then it is called a **kite**.



In quadrilateral ABCD, two pairs of adjacent sides are equal i.e., $AB = AD$ and $BC = DC$

Therefore, ABCD is a kite.

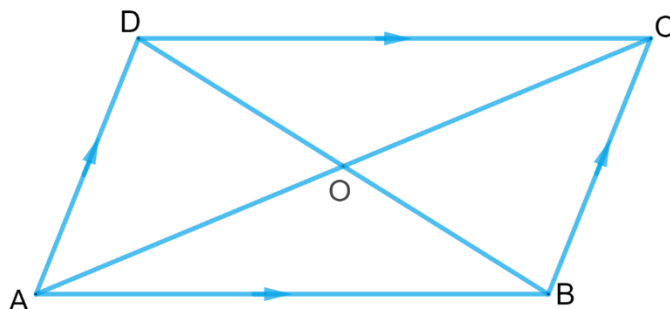
Parallelogram:

A quadrilateral is a parallelogram, if

- (i) opposite sides are equal or
- (ii) opposite angles are equal or
- (iii) diagonals bisect each other or
- (iv) a pair of opposite sides is equal and parallel

In a parallelogram,

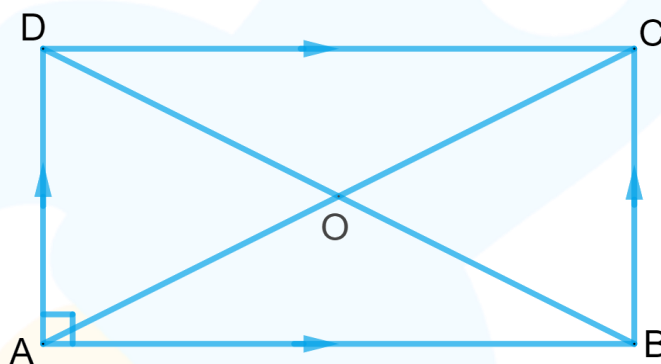
- (i) opposite sides are equal
- (ii) opposite angles are equal
- (iii) diagonals bisect each other



In parallelogram ABCD

- (i) opposite sides are parallel and equal i.e., $AB = DC$ and $BO = OD$
- (ii) opposite angles are equal i.e., $\angle A = \angle C$ and $\angle B = \angle D$
- (iii) diagonals AC and BD bisect each other at point O i.e., $AO = OC$ and $BO = OD$

If in a parallelogram one of its angles is a right angle, then it is called a **rectangle**.



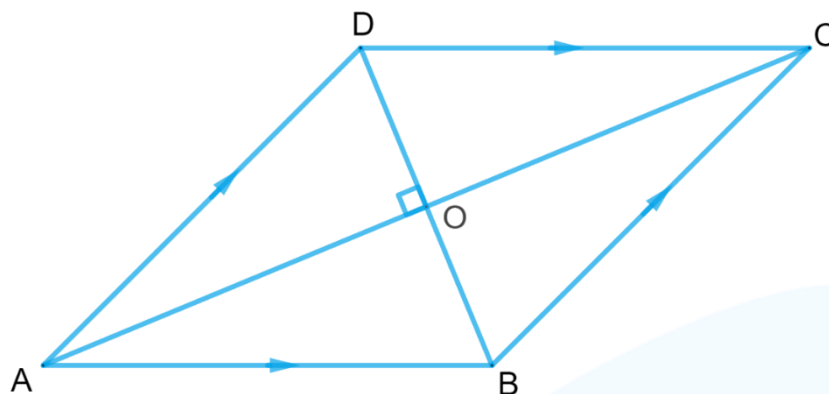
In parallelogram ABCD, $\angle A = 90^\circ$

Therefore, ABCD is a rectangle and

- (i) all angles are equal i.e., $\angle A = \angle B = \angle C = \angle D = 90^\circ$
- (ii) diagonals are equal i.e., $AC = BD$ and $AO = OC = BO = OD$

\

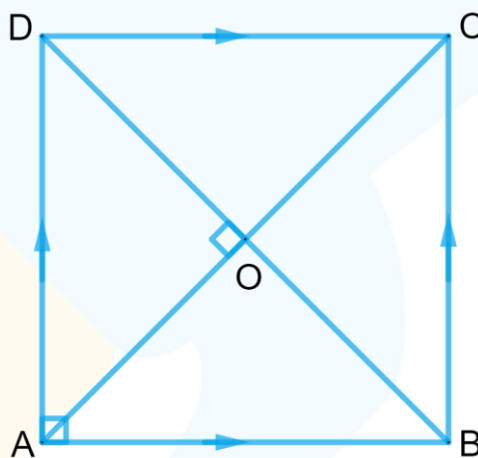
If in a parallelogram all sides are equal, then it is called a **rhombus**.



In parallelogram ABCD, $AB = BC = CD = DA$

Therefore, ABCD is a rhombus and diagonals AC and BD bisect each other perpendicularly i.e., $AO = OC$, $BO = OD$ and $\angle AOB = \angle BOC = \angle COD = \angle DOA = 90^\circ$.

If in a parallelogram one of its angles is a right angle and all sides are equal, then it is called a **square**.



In parallelogram ABCD, $AB = BC = CD = DA$ and $\angle A = 90^\circ$

Therefore, ABCD is a square and

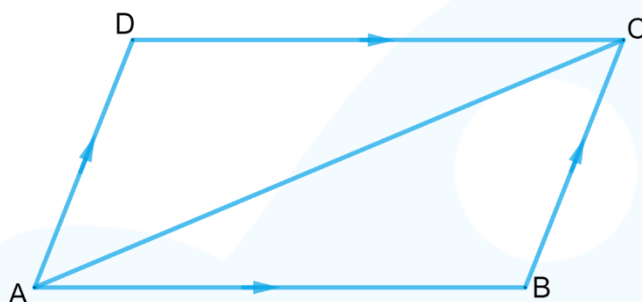
- (i) all angles are equal i.e., $\angle A = \angle B = \angle C = \angle D = 90^\circ$
- (ii) diagonals AC and BD are equal and bisect each other perpendicularly i.e., $AC = BD$ and $AO = OC = BO = OD$, also $\angle AOB = \angle BOC = \angle COD = \angle DOA = 90^\circ$

NOTE:

- A square, rectangle and rhombus are all parallelograms.
- A square is a rectangle and a rhombus as well, but a rectangle or a rhombus is not a square.
- A parallelogram is a trapezium, but a trapezium is not a parallelogram.
- A trapezium with equal non-parallel sides is called isosceles trapezium.

Theorem 1:

A diagonal of a parallelogram divides it into two congruent triangles.



Let ABCD be a parallelogram and AC be a diagonal which divides parallelogram ABCD into two triangles $\triangle ABC$ and $\triangle CDA$

Then $\triangle ABC \cong \triangle CDA$

Theorem 2:

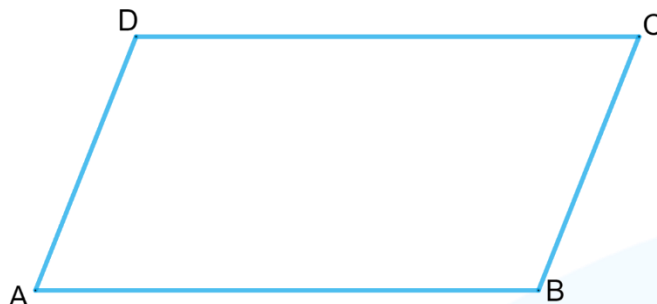
In a parallelogram, opposite sides are equal.



In parallelogram ABCD opposite sides are equal i.e., $AB = CD$ and $AD = BC$

Theorem 3:

If each pair of opposite sides of a quadrilateral is equal, then it is a parallelogram.



In quadrilateral ABCD, opposite sides are equal i.e., $AB = CD$ and $AD = BC$

Then ABCD is a parallelogram.

Theorem 4:

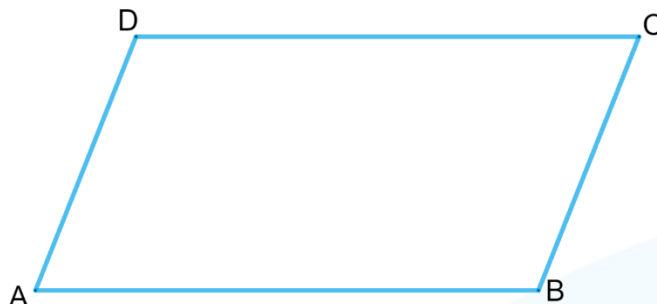
In a parallelogram, opposite angles are equal.



In parallelogram ABCD opposite angles are equal i.e., $\angle A = \angle C$ and $\angle B = \angle D$

Theorem 5:

If in a quadrilateral, each pair of opposite angles is equal, then it is a parallelogram.

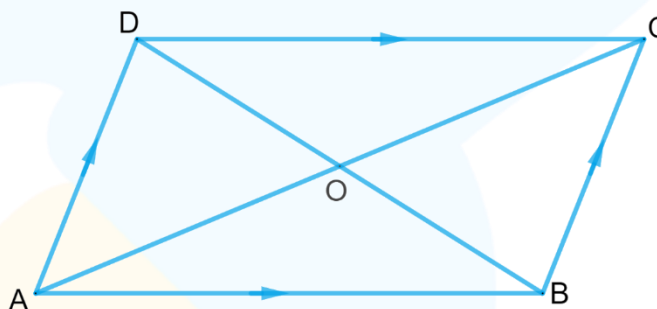


In quadrilateral ABCD, if each pair of opposite angles is equal i.e., $\angle A = \angle C$ and $\angle B = \angle D$

Then ABCD is a parallelogram.

Theorem 6:

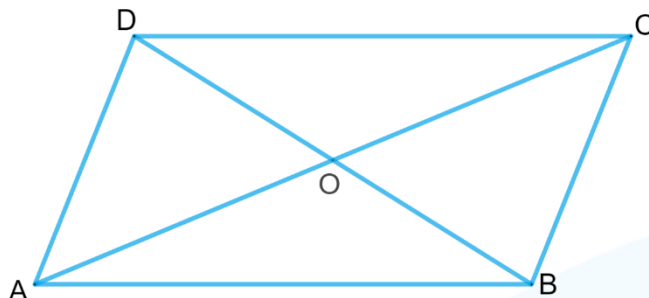
The diagonals of a parallelogram bisect each other.



In parallelogram ABCD diagonals AC and BD bisect each other at point O, i.e., $AO = OC$ and $BO = OD$

Theorem 7:

If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.



In quadrilateral ABCD, diagonals AC and BD bisect each other at O i.e., $AO = OC$ and $BO = OD$.
Then ABCD is a parallelogram.

Theorem 8:

A quadrilateral is a parallelogram if a pair of opposite sides is equal and parallel.



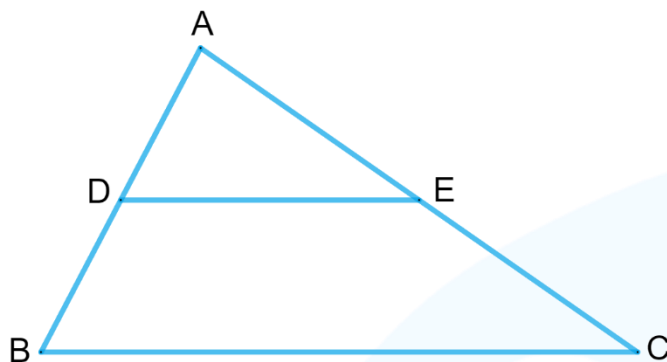
In quadrilateral ABCD, a pair of opposite sides AB and CD is equal and parallel i.e., $AB = CD$ and
 $AB \parallel CD$

Then ABCD is a parallelogram.

The Mid-point Theorem:

Theorem 1

The line segment joining the mid-points of two sides of a triangle is parallel to the third side.

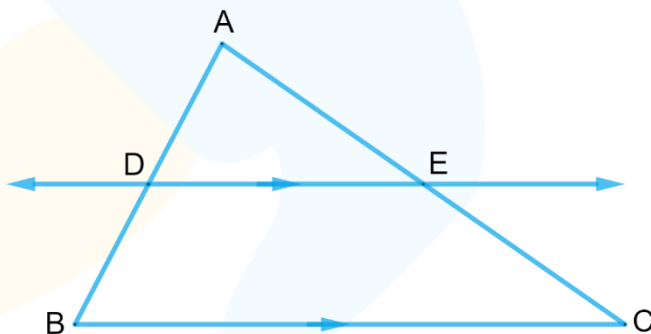


In $\triangle ABC$, D and E are mid-points of AB and AC respectively.

Then $DE \parallel BC$

Theorem 2:

The line drawn through the midpoint of one side of a triangle, parallel to another side bisects the third side.

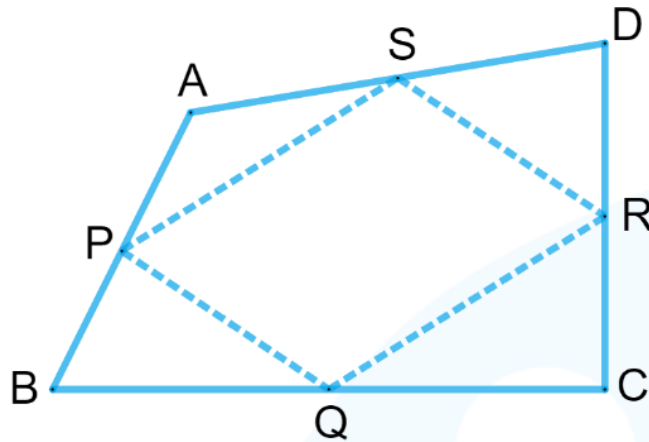


In $\triangle ABC$, DE drawn parallel to BC through the midpoint D of side AB, i.e., $DE \parallel BC$ intersects AC at point E.

Then E is the midpoint of AC, i.e., $AE = CE$

RESULT:

The quadrilateral formed by joining the mid-points of the sides of a quadrilateral, in order, is a parallelogram.

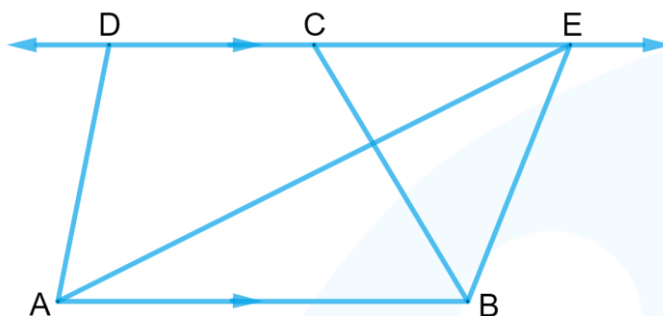


In quadrilateral ABCD, points P, Q, R, and S are mid-points of sides AB, BC, CD and DA respectively. Then PQRS is a parallelogram.

AREAS OF PARALLELOGRAMS AND TRIANGLES

A Parallelogram on the same base and between the same Parallels:

Two figures are said to be on the same base and between the same parallels, if they have a common base (side) and the vertices (or the vertex) opposite to the common base of each figure lie on a line parallel to the base.



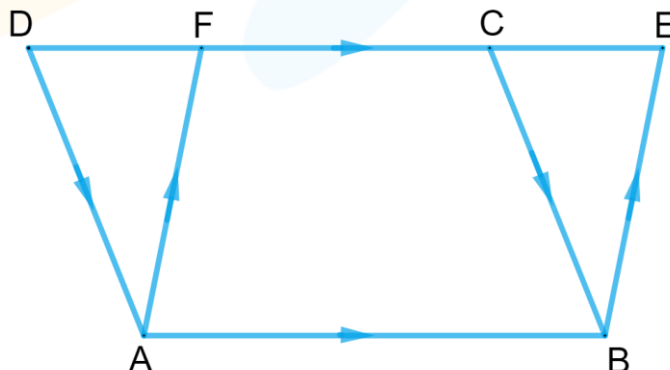
In the above given figure, quadrilateral $ABCD$ and triangle $\triangle ABE$ are on the same base AB and between the same parallels $AB \parallel DE$.

NOTE:

- If two figures are congruent, they must have equal areas.
- Two figures having equal areas need not be congruent.

Theorem 1:

Parallelograms on the same base and between the same parallels are equal in area.



In the above given figure, parallelograms $ABCD$ and $ABEF$ are on the same base AB and between the same parallels $AB \parallel DE$ and $ar(\square ABCD) = ar(\square ABEF)$.

Let's prove this theorem.

Given that $ABCD$ and $ABEF$ are parallelograms.

$AD \parallel BC$ and DE is transversal, then corresponding angles;

$$\angle ADF = \angle BCE$$

$AF \parallel BE$ and DE is transversal, then corresponding angles;

$$\angle AFD = \angle BEC$$

By Angle sum property of a triangle

$$\angle DAF = \angle CBE$$

Now, in $\triangle ADF$ and $\triangle BCE$,

$$AD = BC \quad (\text{opposite sides of the parallelogram } ABCD)$$

$$AF = BE \quad (\text{opposite sides of the parallelogram } ABEF)$$

$$\angle DAF = \angle CBE$$

By SAS criterion of congruence

$$\triangle ADF \cong \triangle BCE$$

Congruent triangles have equal areas

Therefore, $ar(\triangle ADF) = ar(\triangle BCE)$

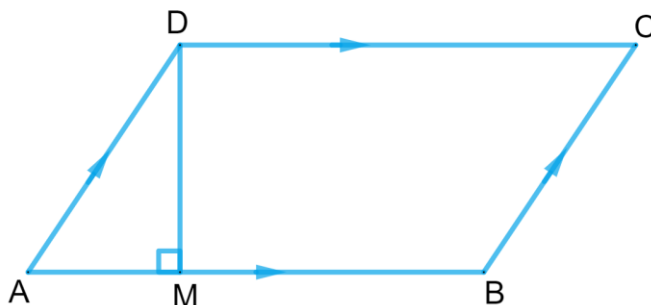
Now,

$$\begin{aligned}
 ar(\square ABCD) &= ar(\triangle ADF) + ar(\square ADFC) \\
 &= ar(\triangle BCE) + ar(\square ADFC) \\
 &= ar(\square ABEF)
 \end{aligned}$$

So, parallelograms $ABCD$ and $ABEF$ are equal in area.

NOTE:

Area of a parallelogram is the product of its any side and the corresponding altitude.

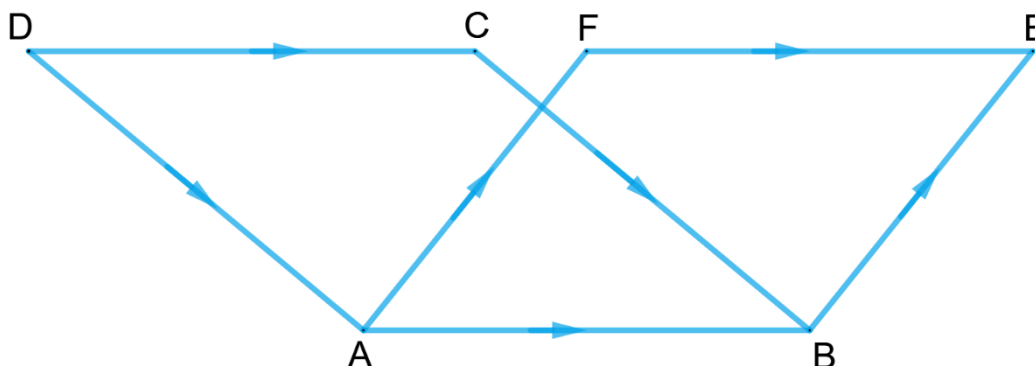


In parallelogram $ABCD$, DM is the altitude to the corresponding side AB .

So, $ar(\square ABCD) = AB \times DM$.

Converse of Theorem 1:

Parallelograms on the same base (or equal bases) and having equal areas lie between the same parallels.

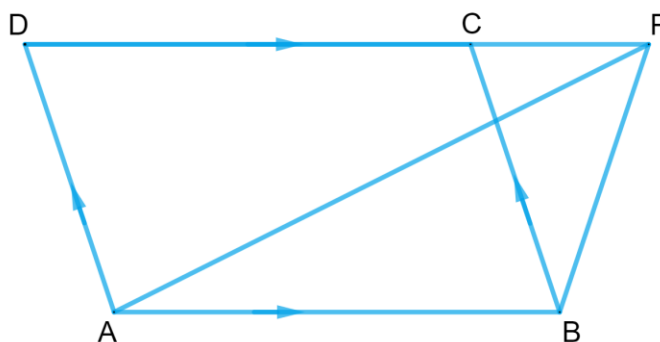


In the figure, parallelograms $ABCD$ and $ABEF$ are on the same base AB and have equal areas i.e., $ar(\square ABCD) = ar(\square ABEF)$.

Then both the parallelograms will lie between the same parallels $AB \parallel DE$, i.e., D, C, F and E are collinear.

NOTE:

If a parallelogram and a triangle are on the same base and between the same parallels, then the area of the triangle is half the area of the parallelogram.



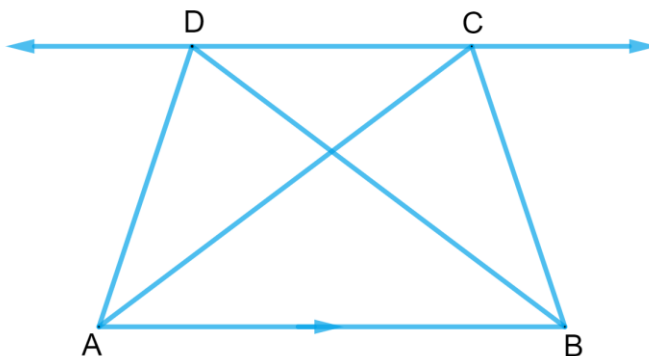
In the figure parallelogram $ABCD$ and triangle $\triangle ABP$ are on the same base AB and between the same parallels $AB \parallel DP$

Then the area of the triangle is half the area of the parallelogram.

i.e.,
$$ar(\triangle ABP) = \frac{1}{2} \times ar(\square ABCD)$$

Triangles on the same base and between the same Parallels:

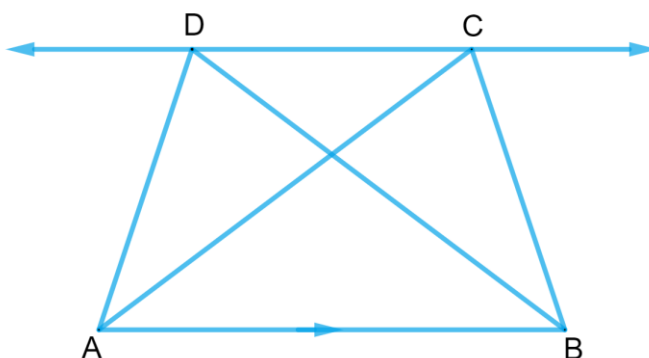
Two triangles are said to be on the same base and between the same parallels, if they have a common base and the vertex opposite to the common base of each triangle lie on a line parallel to the base.



In the figure triangles $\triangle ABC$ and $\triangle ABD$ are on the same base AB and between the same parallels $AB \parallel DC$.

Theorem 2:

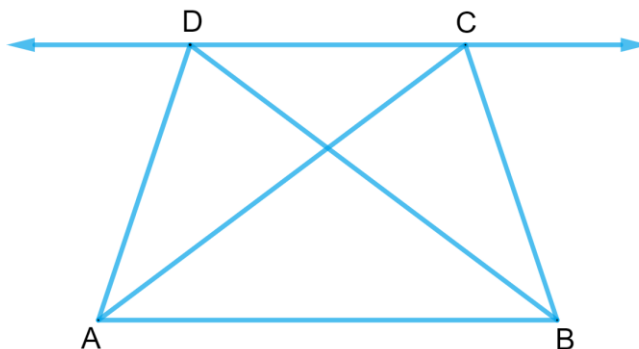
Two triangles on the same base (or equal bases) and between the same parallels are equal in area.



In the figure triangles $\triangle ABC$ and $\triangle ABD$ are on the same base AB and between the same parallels $AB \parallel DC$ and $ar(\triangle ABC) = ar(\triangle ABD)$

Theorem 3:

Two triangles having the same base (or equal bases) and equal areas lie between the same parallels.

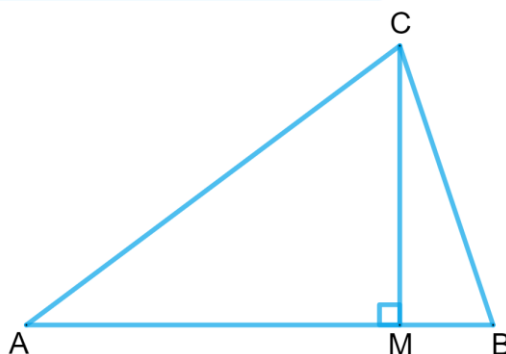


In the figure, triangles $\triangle ABC$ and $\triangle ABD$ are on the same base AB and have equal areas, i.e., $ar(\triangle ABC) = ar(\triangle ABD)$.

Then both the triangles lie between the same parallels i.e., $AB \parallel DC$.

NOTE:

Area of a triangle is half the product of its base and the corresponding altitude.

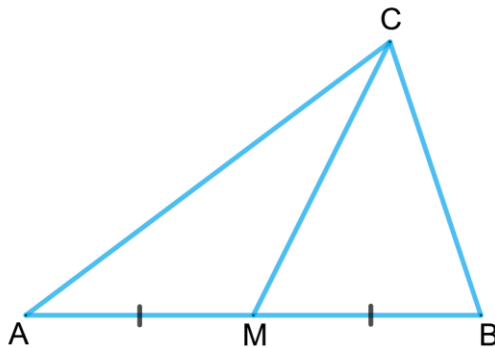


In triangle $\triangle ABC$, AB is the base and CM is the corresponding altitude.

Then, $ar(\triangle ABC) = \frac{1}{2} \times AB \times CM$.

NOTE:

A median of a triangle divides it into two triangles of equal areas.



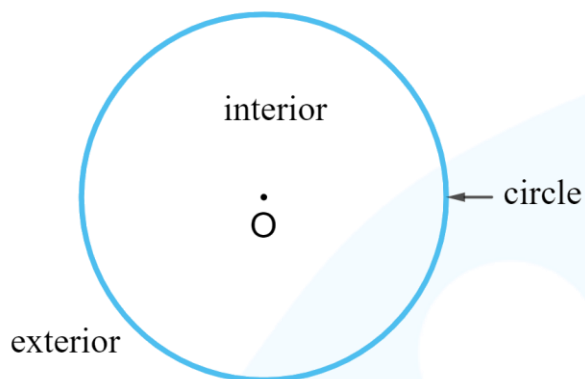
In $\triangle ABC$, CM is a median to the base AB and dividing it into two triangles of equal areas.

i.e., $ar(\triangle ACM) = ar(\triangle BCM)$.

CIRCLES

Circle:

The collection of all the points in a plane, which are at a fixed distance from a fixed point (center) in the plane, is called a circle.



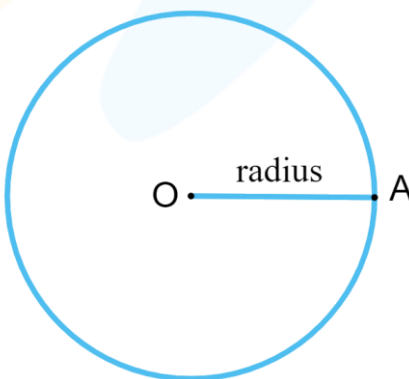
A circle divides the plane on which it lies into three parts. They are:

- (i) inside the circle, which is also called the interior of the circle
- (ii) the circle and
- (iii) outside the circle, which is also called the exterior of the circle

The circle and its interior make up the circular region.

Radius:

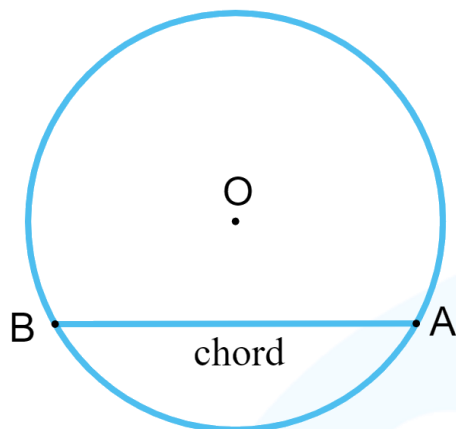
The line segment joining the center and any point on the circle is also called a radius of the circle. It is denoted by r .



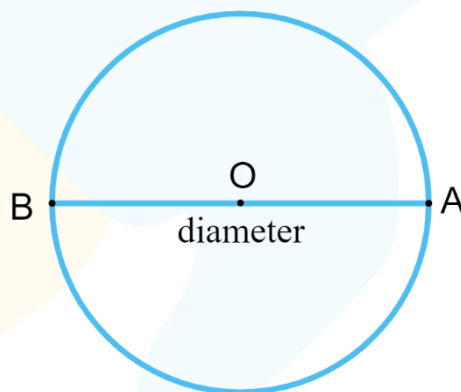
NOTE: A circle with center O and radius r is denoted by $C(O, r)$.

Chord:

The line segment joining any two points on a circle is called a chord of the circle.

**Diameter:**

The chord, which passes through the center of the circle, is called the diameter of the circle. Diameter is the longest chord and all diameters have the same length, which is equal to two times the radius. It is denoted by d .

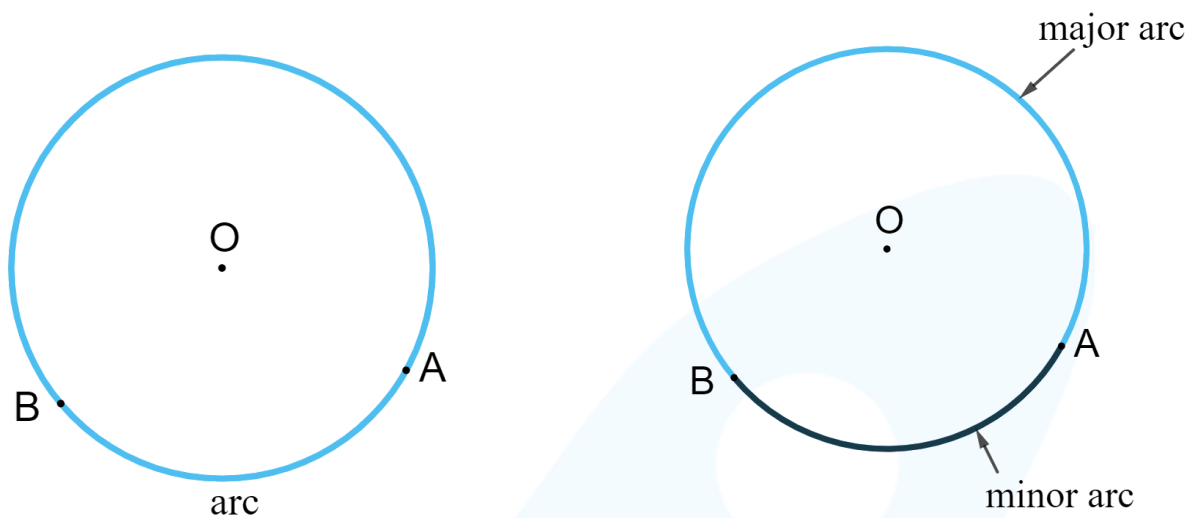


In the figure, AB is the diameter

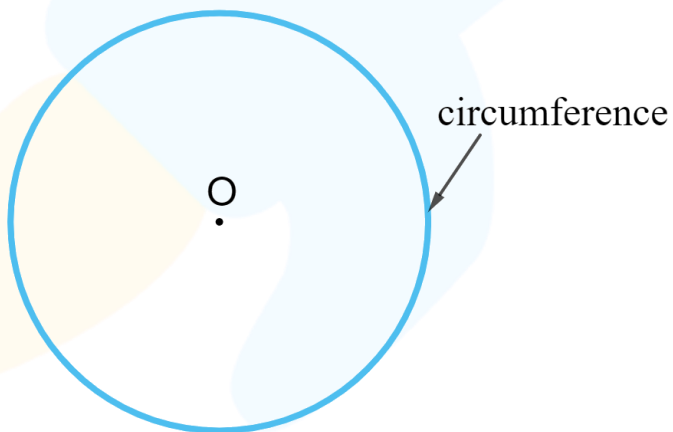
$$\begin{aligned} AB &= OA + OB \\ &= r + r \\ d &= 2r \end{aligned}$$

Arc:

A piece of a circle between two points is called an arc. The longer one is called the major arc and the shorter one is called the minor arc.

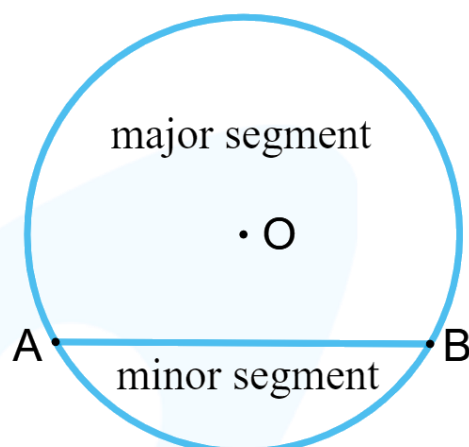
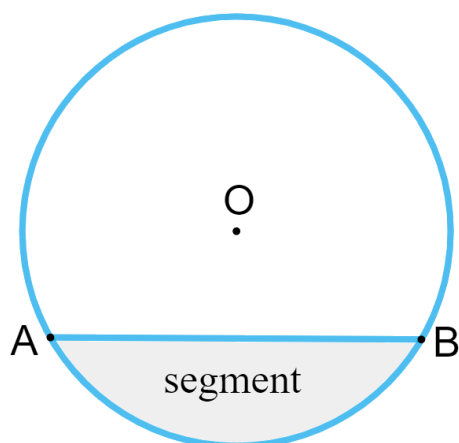
**Circumference:**

The length of the complete circle is called its circumference.

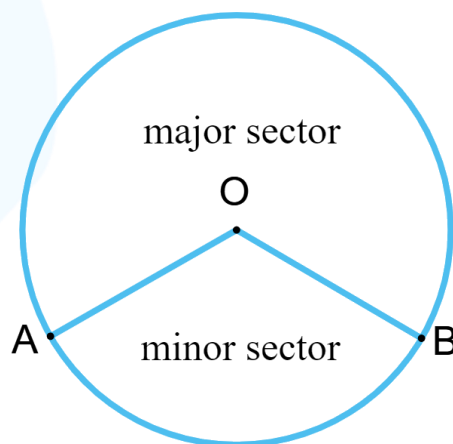
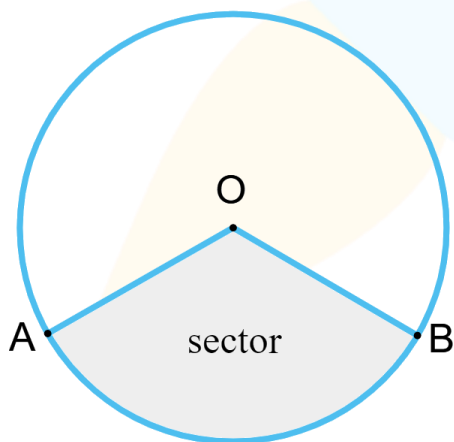


Segment:

The region between a chord and either of its arcs is called a segment of the circular region or simply a segment of the circle. There are two types of segments, which are the major segment and the minor segment

**Sector:**

The region between an arc and the two radii, joining the center to the endpoints of the arc is called a sector. The minor arc corresponds to the minor sector and the major arc corresponds to the major sector. There are two types of sectors, which are the major sector and the minor sector.

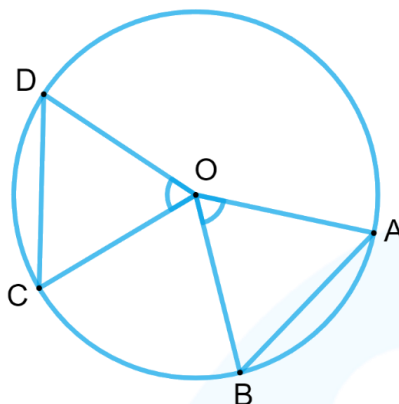
**NOTE:**

When two arcs are equal, that is, each is a semicircle, then both segments and both sectors become the same and each is known as a semicircular region.

Angle Subtended by a Chord at a Point:

Theorem 1:

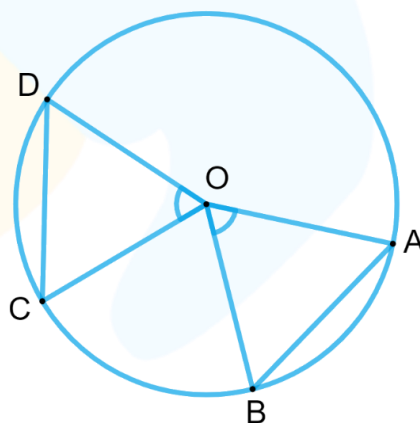
Equal chords of a circle subtend equal angles at the center.



In circle, $C(O, r)$, two equal chords AB and CD , i.e., $AB = CD$ are subtending equal angles, $\angle AOB$ and $\angle COD$ at the centers respectively, i.e., $\angle AOB = \angle COD$.

Theorem 2:

If the angles subtended by the chords of a circle at the center are equal, then the chords are equal.



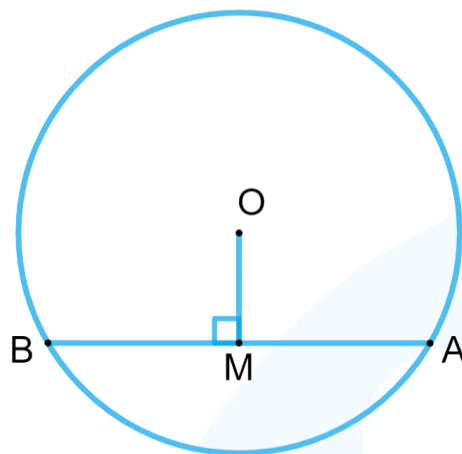
In circle, $C(O, r)$, two chords AB and CD are subtending equal angles, $\angle AOB$ and $\angle COD$ at the centers respectively, i.e., $\angle AOB = \angle COD$.

Then the chords AB and CD are equal, i.e., $AB = CD$.

Perpendicular from the Center to a Chord:

Theorem 3:

The perpendicular from the center of a circle to a chord bisects the chord.

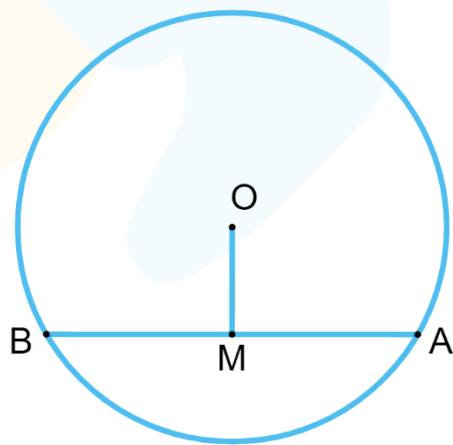


In circle $C(O, r)$ from the center a perpendicular OM is drawn to the chord AB , i.e., $OM \perp AB$.

Then the perpendicular OM bisects the chord AB at point M , i.e., $AM = BM$.

Theorem 4:

The line drawn through the center of a circle to bisect a chord is perpendicular to the chord.

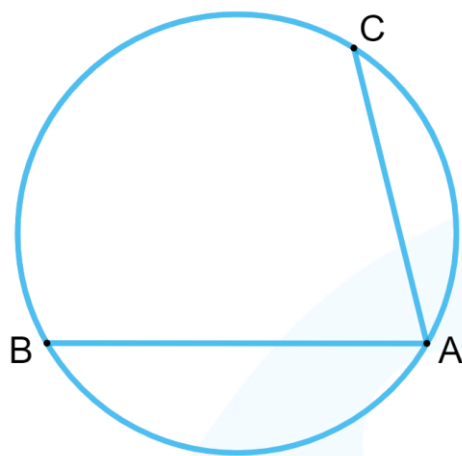


In circle, $C(O, r)$, a line OM is drawn from the center to the chord AB which bisects the chord at point M , i.e., $AM = BM$. Then the OM is perpendicular to AB i.e., $OM \perp AB$.

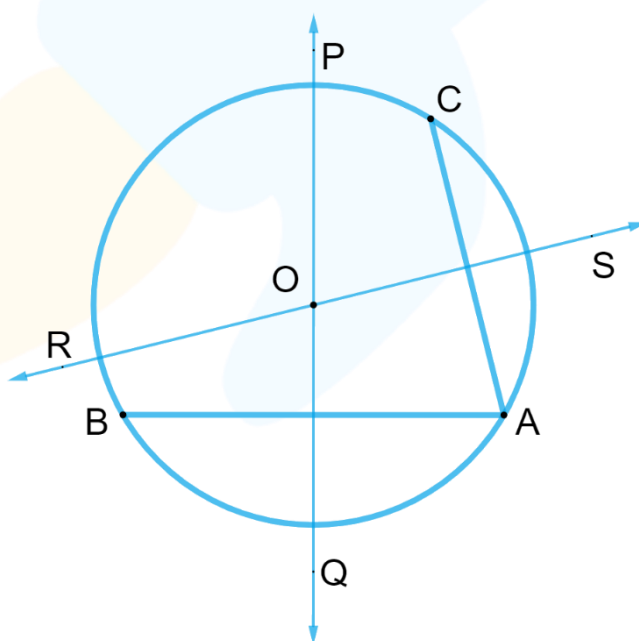
Circle through Three Points:

Theorem 5:

There is one and only one circle passing through three given non-collinear points.



Let us take three points A , B and C , which are not on the same line, i.e., they are non-collinear. Draw perpendicular bisectors of AB and AC , PQ and RS respectively. Let these perpendicular bisectors intersect at one-point O .



Now O lies on the perpendicular bisector PQ of AB , we have $OA=OB$, as every point on the perpendicular bisector of a line segment is equidistant from its end points.

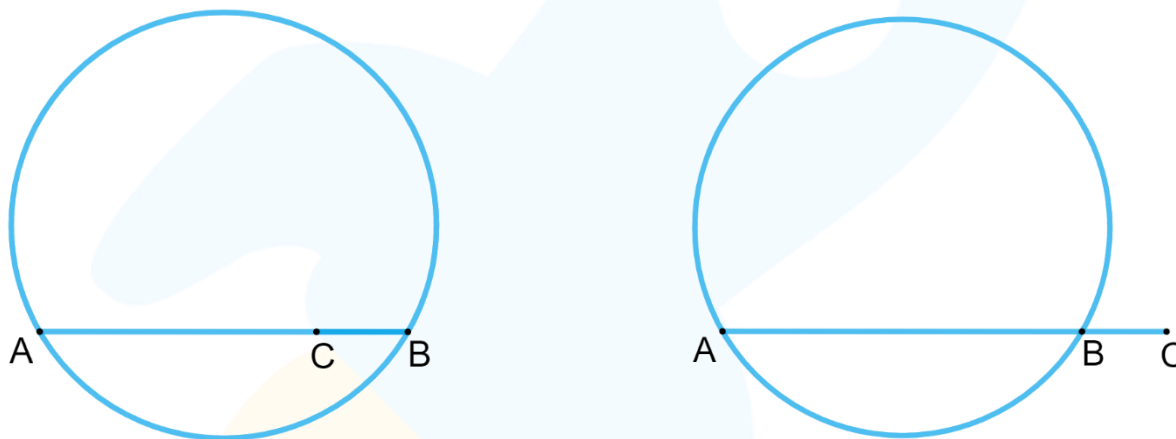
Similarly, as O lies on the perpendicular bisector RS of AC , we get $OB = OC$

So, $OA = OB = OC$, which means that the points A , B and C are equidistant from the point O , and if we draw a circle with center O and radius OA , it will also pass through B and C . This shows that there is a circle passing through the three points A , B and C .

We know that two lines can intersect at only one point, so we can draw only one circle with radius OA .

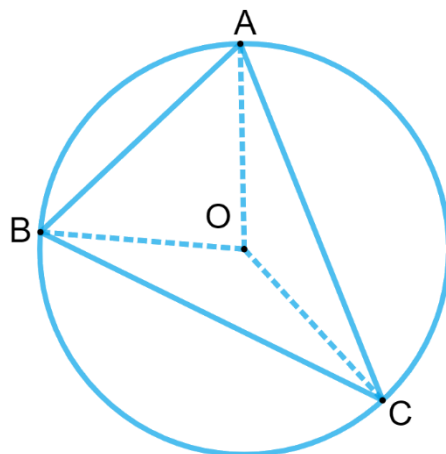
In other words, there is a unique circle passing through A , B and C .

If the points A , B and C lie on a line, i.e., collinear, then the third point will lie inside or outside the circle passing through two points.



NOTE:

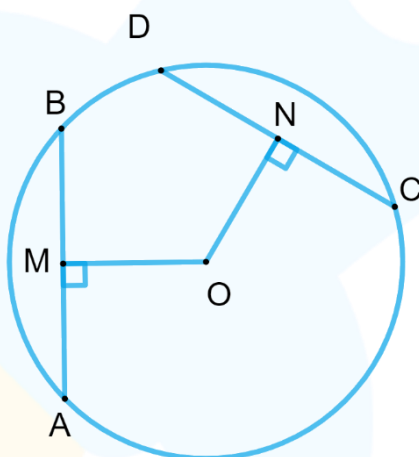
If ABC is a triangle, then there is a unique circle passing through the three vertices A , B and C of the triangle. This circle is called the **circumcircle** of the $\triangle ABC$. Its center and radius are called respectively the **circumcenter** and the **circumradius** of the triangle.



Equal Chords and Their Distances from the Center:

Theorem 6:

Equal chords of a circle (or of congruent circles) are equidistant from the center (or centers).

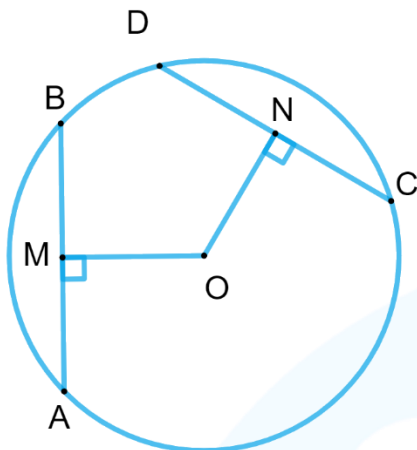


In circle $C(O, r)$ two equal chords AB and CD , i.e., $AB = CD$. Distances of AB and CD from the center are OM and ON respectively.

Then, $OM = ON$.

Theorem 7:

Chords equidistant from the center of a circle are equal in length.



In circle $C(O, r)$, OM and ON are equal distances of two chords AB and CD from the center O respectively, i.e., $OM = ON$. Then chords AB and CD are equal in length i.e., $AB = CD$.

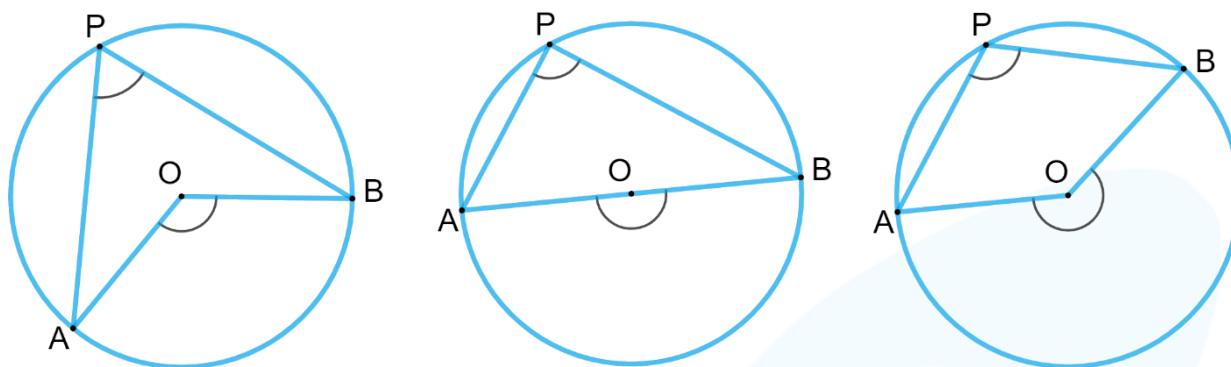
Angle Subtended by an Arc of a Circle:

If two chords of a circle are equal, then their corresponding arcs are congruent and conversely, if two arcs are congruent, then their corresponding chords are equal.

Congruent arcs (or equal arcs) of a circle subtend equal angles at the center.

Theorem 8:

The angle subtended by an arc at the center is double the angle subtended by it at any point on the remaining part of the circle.

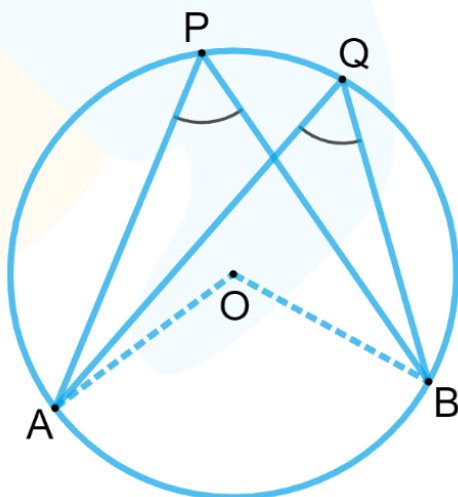


In circle $C(O, r)$ an arc AB subtending angles $\angle AOB$ at center O and $\angle APB$ at a point P on the remaining part of the circle.

Then, $\angle AOB = 2\angle APB$.

Theorem 9:

Angles in the same segment of a circle are equal.

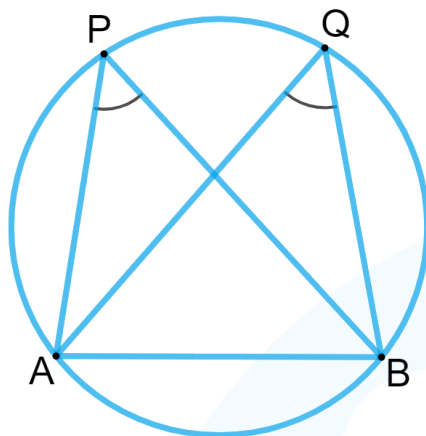


In circle $C(O, r)$ an arc AB subtending angles $\angle APB$ and $\angle AQB$ at points P and Q in the same segment of the circle respectively.

Then, $\angle APB = \angle AQB$.

Theorem 10:

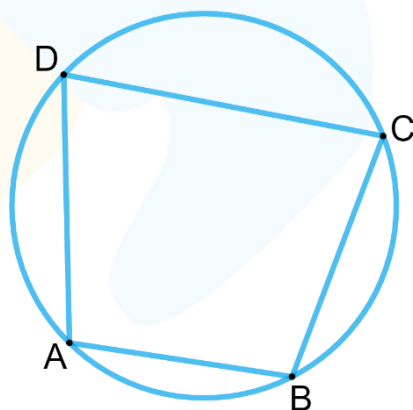
If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, the four points lie on a circle (i.e. they are concyclic).



AB is a line segment, which subtends equal angles at two points P and Q . That is $\angle APB = \angle AQB$. Then, the four points A, B, P and Q lie on a circle i.e. they are concyclic.

Cyclic Quadrilaterals:

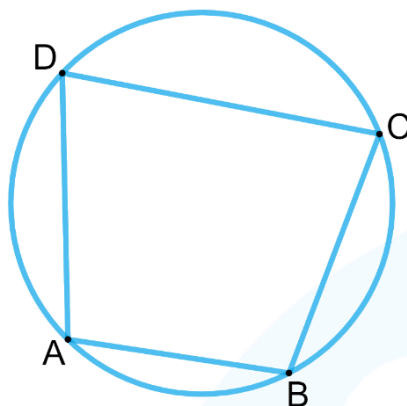
A quadrilateral $ABCD$ is called cyclic if all the four vertices of it lie on a circle.



In the figure $ABCD$ is a cyclic quadrilateral.

Theorem 11:

The sum of either pair of opposite angles of a cyclic quadrilateral is 180° .



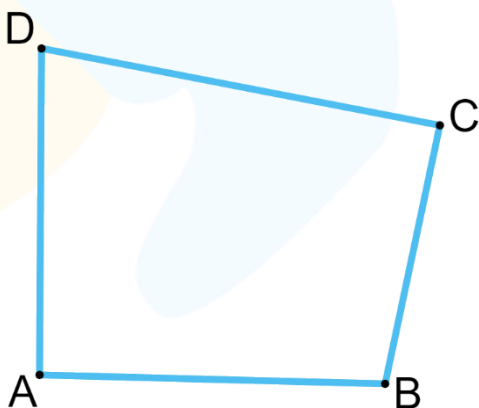
In cyclic quadrilateral $ABCD$ pairs of opposite angles are supplementary, i.e.,

$$\angle A + \angle C = 180^\circ$$

$$\angle B + \angle D = 180^\circ$$

Theorem 12:

If the sum of a pair of opposite angles of a quadrilateral is 180° , the quadrilateral is cyclic.



In the quadrilateral $ABCD$ if pairs of opposite angles are supplementary, i.e., $\angle A + \angle C = 180^\circ$ and $\angle B + \angle D = 180^\circ$.

Then $ABCD$ is a cyclic quadrilateral.

HERON'S FORMULA

Heron's Formula:

This formula is given by Heron about the area of a triangle and known as Heron's formula. It is stated as:

$$\text{Area of a triangle} = \sqrt{s(s-a)(s-b)(s-c)}$$

where a , b and c are the sides of the triangle, and 's' is semi-perimeter, i.e., half the perimeter of the triangle i.e., $s = \frac{a+b+c}{2}$

This formula is helpful where it is not possible to find the height of the triangle and all three sides are known.

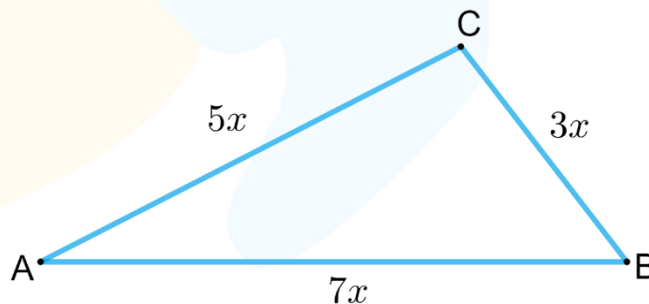
Let us use this formula to find the area of a triangle:

Example:

The sides of a triangular plot are in the ratio of 3:5:7 and its perimeter is 300m. Find its area.

Solution:

Suppose that the sides are $3x$, $5x$ and $7x$ in metres.



Perimeter of the triangle = 300m

$$3x + 5x + 7x = 300$$

$$15x = 300$$

$$x = \frac{300}{15}$$

$$x = 20$$

Therefore, $3x = 3 \times 20m = 60m$

$$5x = 5 \times 20m = 100m$$

$$7x = 7 \times 20m = 140m$$

Hence the sides of the triangle are $60m, 100m$ and $140m$.

Now use Heron's formula,

$$a = 60m, b = 100m, c = 140m$$

$$s = \frac{60m + 100m + 140m}{2} = \frac{300m}{2} = 150m$$

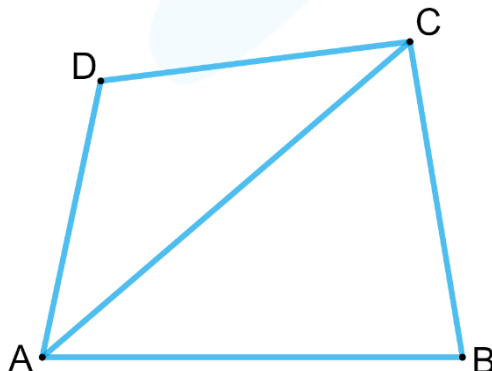
$$\begin{aligned}\text{Area of the triangle} &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{150m(150m-60m)(150m-100m)(150m-140m)} \\ &= \sqrt{150m \times 90m \times 50m \times 10m} \\ &= 1500\sqrt{3}m^2\end{aligned}$$

Therefore, area of the triangle is $1500\sqrt{3}m^2$.

Heron's Formula in Finding Areas of Quadrilaterals:

Area of a quadrilateral whose sides and one diagonal are given, can be calculated by dividing the quadrilateral into two triangles and using the Heron's formula.

Let us draw diagonal AC in a quadrilateral $ABCD$.



If length of the all four sides AB, BC, CD, DA and diagonal AC is known, then

Area of quadrilateral $ABCD = ar(\triangle ABC) + ar(\triangle ADC)$

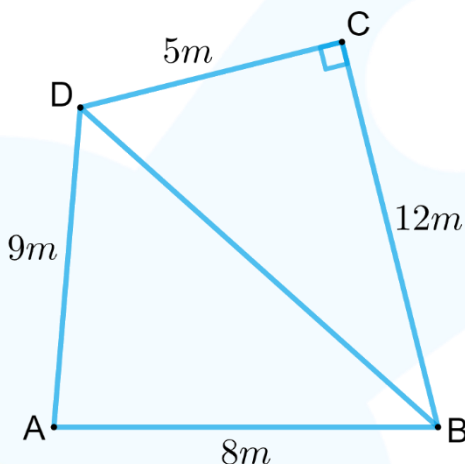
Let us use Heron's formula to find the area of a quadrilateral whose sides and one diagonal are given by dividing it into two triangles.

Example:

A park, in the shape of a quadrilateral $ABCD$, has $AB = 8\text{ m}$, $BC = 12\text{ m}$, $CD = 5\text{ m}$, $DA = 9\text{ m}$ and $\angle C = 90^\circ$. How much area does it occupy?

Solution:

Let us draw quadrilateral $ABCD$ and join diagonal BD .



Now we can see that $\triangle BCD$ is a right triangle in which $BC = 12\text{ m}$, $CD = 5\text{ m}$ and $\angle C = 90^\circ$

By using Pythagoras theorem

$$\begin{aligned} BD^2 &= BC^2 + CD^2 \\ BD &= \sqrt{BC^2 + CD^2} \\ &= \sqrt{(12\text{m})^2 + (5\text{m})^2} \\ &= \sqrt{144\text{m}^2 + 25\text{m}^2} \\ &= \sqrt{169\text{m}^2} \\ &= 13\text{m} \end{aligned}$$

Now, area of a triangle $= \frac{1}{2} \times \text{base} \times \text{height}$

$$\begin{aligned} \text{ar}(\triangle BCD) &= \frac{1}{2} \times BC \times CD \\ &= \frac{1}{2} \times 12m \times 5m \\ &= 30m^2 \end{aligned}$$

Now in $\triangle ABD$, $AB = 8m$, $BD = 13m$, $DA = 9m$

Now use Heron's formula,

$$a = 8m, b = 13m, c = 9m$$

$$s = \frac{8m + 13m + 9m}{2} = \frac{30m}{2} = 15m$$

$$\begin{aligned} \text{Area of the triangle} &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{15m(15m-8m)(15m-13m)(15m-9m)} \\ &= \sqrt{15m \times 7m \times 2m \times 6m} \\ &= 6\sqrt{35}m^2 \\ &= 6 \times 5.916m^2 \\ &= 35.496m^2 \end{aligned}$$

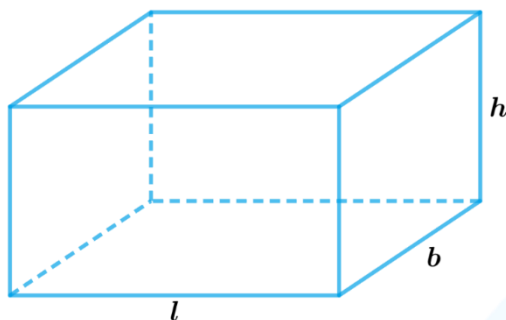
$$\begin{aligned} \text{ar}(\square ABCD) &= \text{ar}(\triangle ABD) + \text{ar}(\triangle BCD) \\ &= 30m^2 + 35.496m^2 \\ &= 65.496m^2 \end{aligned}$$

Therefore, area of quadrilateral $ABCD = 65.496m^2$

SURFACE AREAS AND VOLUMES

Surface Areas and Volumes of Solids:

1) Cuboid:



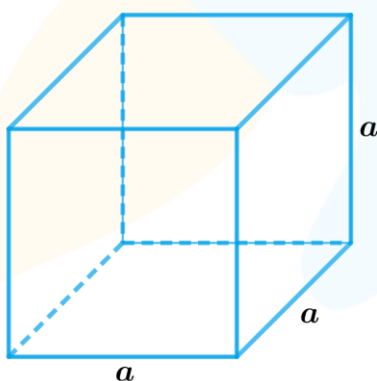
Lateral surface area of cuboid $= 2(l+b)h$

Total surface area of cuboid $= 2(lb+bh+hl)$

Volume of cuboid $= lbh$

Where l , b and h are the length, breadth and height of the cuboid respectively.

2) Cube:



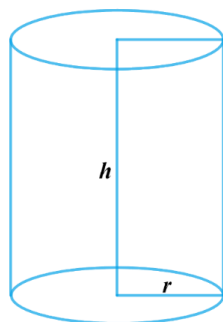
Lateral surface area of cube $= 4a^2$

Total surface area of cube $= 6a^2$

Volume of cube $= a^3$

Where a , is the length of each edge of the cube.

3) Cylinder:



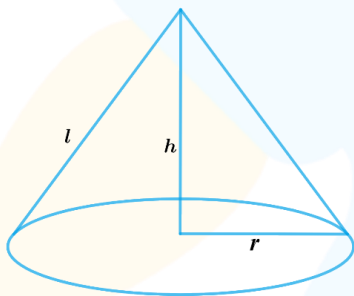
CSA of the right circular cylinder $= 2\pi rh$

TSA of the right circular cylinder $= 2\pi r(h + r)$

Volume of the right circular cylinder $= \pi r^2 h$

Where r and h are the radius and height of the cylinder respectively.

4) Cone:



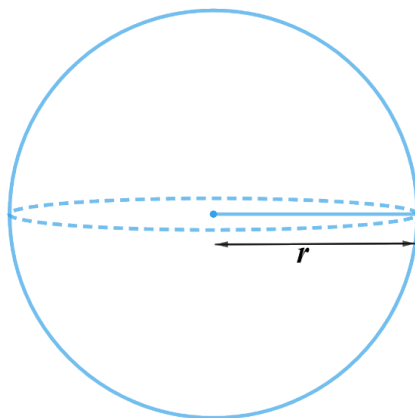
CSA of the right circular cone $= \pi rl$

TSA of the right circular cone $= \pi r(l + r)$

Volume of the right circular cone $= \frac{1}{3} \pi r^2 h$

Slant height of the right circular cone, $l = \sqrt{r^2 + h^2}$

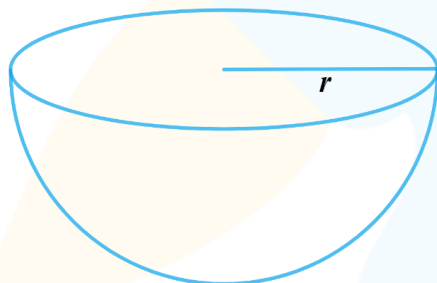
Where r , h and l are the radius, height and slant height of the cone respectively.

5) Sphere:

Surface area of the sphere $= 4\pi r^2$

Volume of the sphere $= \frac{4}{3}\pi r^3$

Where r is the radius of the sphere.

6) Hemisphere:

CSA of the hemisphere $2\pi r^2$

TSA of the hemisphere $3\pi r^2$

Volume of the hemisphere $= \frac{2}{3}\pi r^3$

Where r is the radius of the hemisphere.

STATISTICS

Data:

The facts or figures, which are numerical or otherwise, collected with a definite purpose are called data. Data is the plural form of the Latin word **datum**.

Collection of Data:

When a person collects the information by himself with a definite objective in his mind, the data obtained is called **primary data**.

When a person gathers the information from a source which already had the information stored and collected by someone else in another context, is called **secondary data**.

Secondary data needs to be used with great care ensuring that the source is reliable.

Presentation of Data:

After completing the step of Collection of Data, the Next step is to find out ways to present them in a form which is meaningful, easily understood and gives its main features at a glance.

For example, 10 students in a mathematics test scores as:

85 63 75 77 54 46 29 57 71 44

The data in this form is called **raw data**.

By looking at it in this form, we can easily find the highest and the lowest marks. If these scores are arranged in ascending or descending order, it would be less time-consuming.

So, let us arrange the scores in ascending order as;

29 44 46 54 57 63 71 75 77 85

Now, we can clearly see that the lowest score is 29 and the highest score is 85.

Range: The difference of the highest and the lowest values in the data is called the **range** of the data. So, the range in the above case is $85 - 29 = 56$.

Ungrouped Frequency Distribution Table:

When the number of observations in an experiment is large then the presentation of data in ascending or descending order can be quite time-consuming.

For example, the marks, out of 100, obtained by 30 students of Class IX of a school as:

19 25 37 93 98 40 50 59 60 70
93 85 80 70 72 70 37 40 37 40
93 40 50 50 59 60 70 60 60 85

To make the data more easily understandable, we write it in a table, as given below:

Marks	Tally Marks	Number of students (frequency)
19		1
25		1
37		3
40		4
50		3
59		2
60		4
70		4
72		1
80		1
85		2
93		3
98		1
Total		30

This is called an ungrouped frequency distribution table, or simply a frequency distribution table. We also have used tally marks in preparing these tables.

Grouped Frequency Distribution Table:

For example, In the month of May, television's series watched by 100 children in hours as:

12	23	30	98	45	50	90	60	75	31
98	45	60	60	75	44	36	40	37	42
93	30	30	30	85	60	70	65	60	85
45	40	50	60	79	61	72	19	40	50
98	45	95	50	69	60	70	60	75	45
45	45	50	60	79	16	72	90	40	52
56	44	52	62	59	36	17	19	46	45
42	48	50	65	49	76	73	92	54	88
49	24	60	63	69	66	27	79	67	23
44	54	87	67	91	60	78	93	44	77

To present such a large amount of data so that a reader can make sense of it easily, by taking them into groups like 10 – 19, 20 – 29, . . . , 90 – 99, since our data is from 12 to 98. These groupings are called **classes** or **class-intervals**, and their size is called the **class-size** or **class-width**, which is 10 in this case.

In each of these classes, the least number is called the **lower-class limit** and the greatest number is called the **upper-class limit**, e.g., in 10 – 19, 10 is the lower-class limit and 19 is the upper-class limit.

The data above can be condensed in tabular form as follows:

Marks	Tally Marks	Number of students (frequency)
10 – 19		5
20 – 29		4
30 – 39		8
40 – 49		22
50 – 59		12
60 – 69		21
70 – 79		14
80 – 89		4
90 – 99		10
Total		100

Presenting data in this simplified and condensed form, enables us to observe certain important features at a glance. This is called the **grouped frequency distribution table**.

Here we can easily observe that television's series watched, 60 hours or more by, $21 + 14 + 4 + 10 = 49$ children.

We observe that the classes in the table above are **non-overlapping** or **inclusive**. We could have made more classes of shorter size, or fewer classes of larger size also. So, there is no hard and fast rule about this except that the classes should not overlap.

Now, if a child watches the series for 59.5 hours, then we would not be able to add it in the class ending with 59 or starting with 60.

This is because there are gaps in between the upper and lower limits of two consecutive classes. So, we need to divide the intervals so that the upper and lower limits of consecutive intervals are the same.

For this, we find the difference between the upper limit of a class and the lower limit of its succeeding class.

We then add half of this difference to each of the upper limits and subtract the same from each of the lower limits.

So, let us create the table in which classes are **overlapping** or **exclusive**.

Consider the classes $10 - 19$ and $20 - 29$.

The lower limit of $20 - 29 = 20$

The upper limit of $10 - 19 = 19$

The difference $= 20 - 19 = 1$

So, half the difference $= 0.5$

So, the new class interval formed from $10 - 19$ is $9.5 - 19.5$.

Similarly, the new class formed from the class $20 - 29$ is $19.5 - 29.5$.

Continuing in the same manner, the continuous classes form.

By convention, we consider 59.5 in the class $59.5 - 69.5$ and not in $49.5 - 59.5$.

So, the marks 59.5 would be included in $59.5 - 69.5$.

Now, with these assumptions, the frequency distribution table will be as shown below:

Marks	Tally Marks	Number of students (frequency)
$9.5 - 19.5$		5
$19.5 - 29.5$		4
$29.5 - 39.5$		8
$39.5 - 49.5$		22
$49.5 - 59.5$		12
$59.5 - 69.5$		21
$69.5 - 79.5$		14
$79.5 - 89.5$		4
$89.5 - 99.5$		10
Total		100

Graphical Representation of Data:

The representation becomes easier to understand when individual items are shown graphically than the actual data. Let us discuss the following graphical representations:

- (1) Bar graphs
- (2) Histograms
- (3) Frequency polygons

1. Bar Graphs:

A bar graph is a pictorial representation of data in which bars of uniform width are drawn with equal spacing between them on one axis (usually the x -axis), depicting the variable. The values of the variable are shown on the other axis (usually the y -axis) and the heights of the bars depend on the values of the variable. Usually, bar graphs are drawn for ungrouped frequency distribution tables.

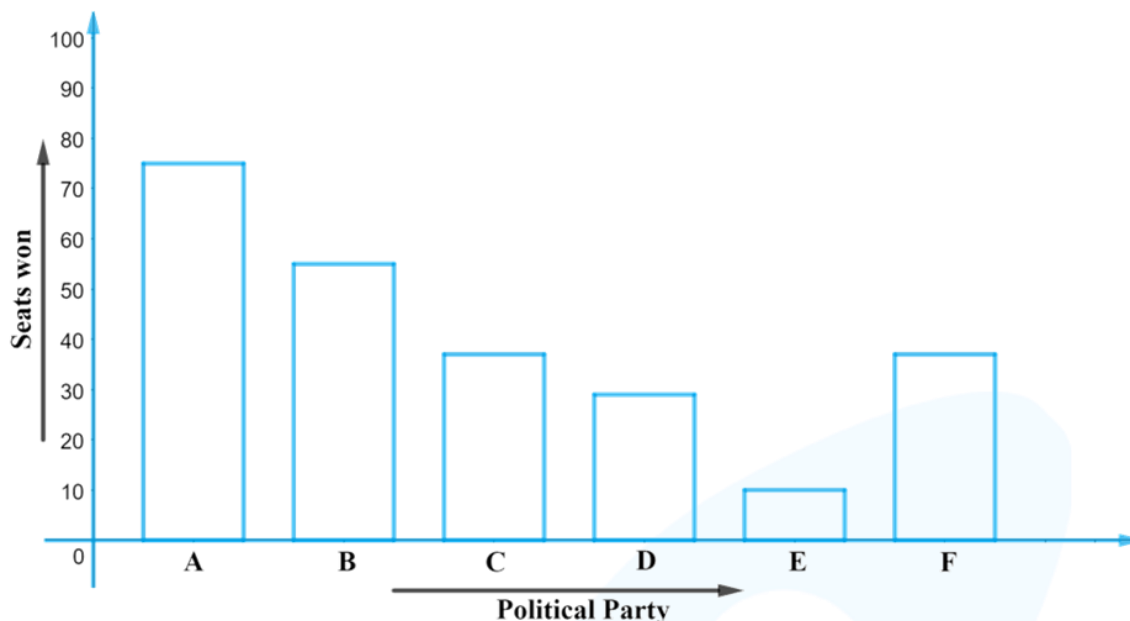
For example, given below are the seats won by different political parties in the polling outcome of a state assembly elections:

Political Party	A	B	C	D	E	F
Seats Won	75	55	37	29	10	37

Let us draw a bar graph for the data above with the following steps:

- 1) We represent the 'Political Party' (variable) on the horizontal axis, choosing any scale, since the width of the bar is not important. But for clarity, we take equal widths for all bars and maintain equal gaps in between.
- 2) We represent the 'Seat Won' (value) on the vertical axis. Since the minimum value is 10 and the maximum value is 75, we can choose the scale as 1 unit = 10.
- 3) To represent our first Political Party, i.e., A, we draw a rectangular bar with width 1 unit and height 7.5 units. Since the value is 75.
- 4) Similarly, other Political parties are represented leaving a gap of 1 unit in between two consecutive bars.

Hence the bar graph is drawn as:



From the graph, it can easily be observed that the political party 'A' won the maximum number of seats and 'E' won the least number of seats.

2. Histograms:

Histogram is also a pictorial representation like the bar graph, but it is used for continuous class-intervals or grouped frequency distribution table. It can be drawn for uniform width, and varying widths as well.

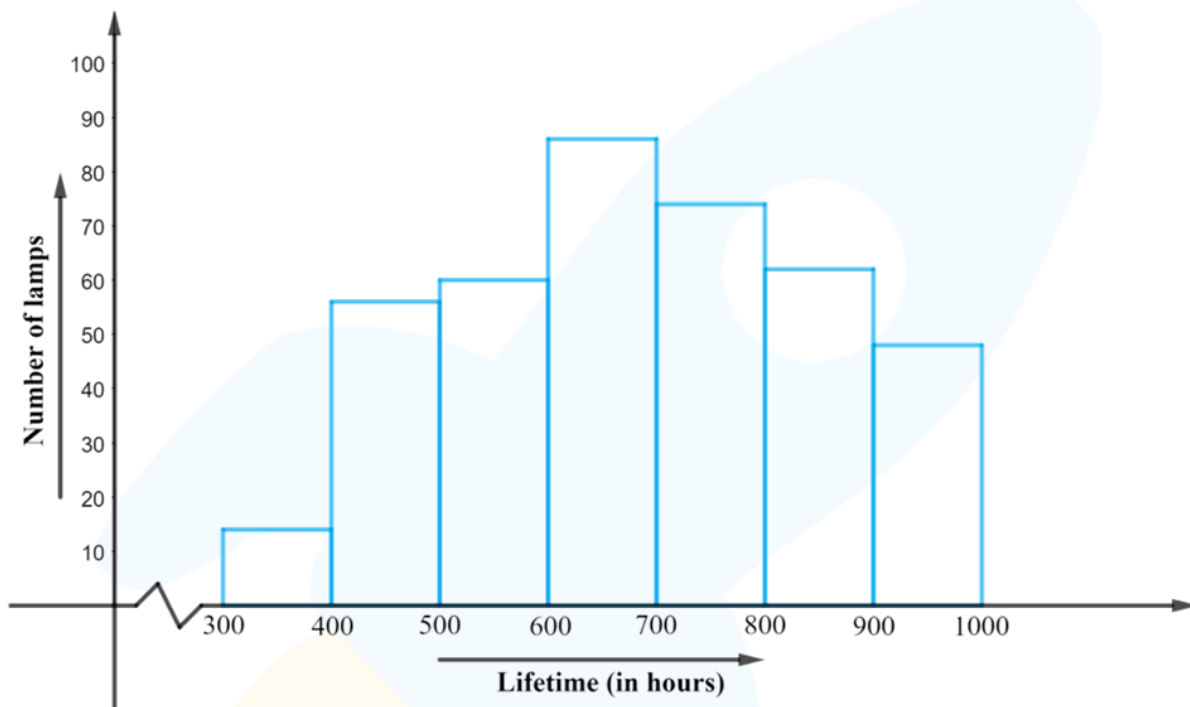
For example, the following table gives the lifetimes of 400 neon lamps:

Lifetime (in hours)	Number of lamps
300 – 400	14
400 – 500	56
500 – 600	60
600 – 700	86
700 – 800	74
800 – 900	62
900 – 1000	48

Let us represent the data given above graphically as follows:

- 1) We represent the weights on the horizontal axis on a suitable scale. Since the first class-interval is starting from 300 and not zero, we show it on the graph by marking a kink or a break on the axis.
- 2) We represent the number of lamps (frequency) on the vertical axis on a suitable scale. Since the minimum frequency is 14 and the maximum frequency is 86, we need to choose the scale to accommodate this maximum frequency.
- 3) We now draw rectangles (or rectangular bars) of width equal to the class-size and lengths according to the frequencies of the corresponding class intervals.

Hence, we obtain the histogram graph as:



We can observe that there are no gaps in between consecutive rectangles as the upper-limit and lower-limit of these two rectangles are the same. This is called a histogram, which is a graphical representation of a grouped frequency distribution with continuous classes as the width of each rectangle is the same. The areas of the rectangles are proportional to the frequencies in this histogram.

Let us take an example where class-intervals vary.

For example, a random survey of the number of children of various age groups playing in a park was found as follows:

Age (in years)	Number of children
1 – 2	5
2 – 3	3
3 – 5	6
5 – 7	12
7 – 10	9
10 – 15	10
15 – 17	4

Let us draw a histogram to represent the data above.

We can observe that the class intervals have varying widths. This will make the rectangular bars to have varying widths and will give us a misleading picture of the data

So, we need to make certain modifications in the length of the rectangles so that the areas of the rectangles are proportional to the frequencies.

We will follow the following steps:

- 1) Select a class interval with the minimum class size. In the example above, the minimum class-size is 1.
- 2) The lengths of the rectangles are then modified to be proportionate to the class-size 1.

For instance, when the class-size is 5, the frequency is 10. So, when the class size is 1, the length of the

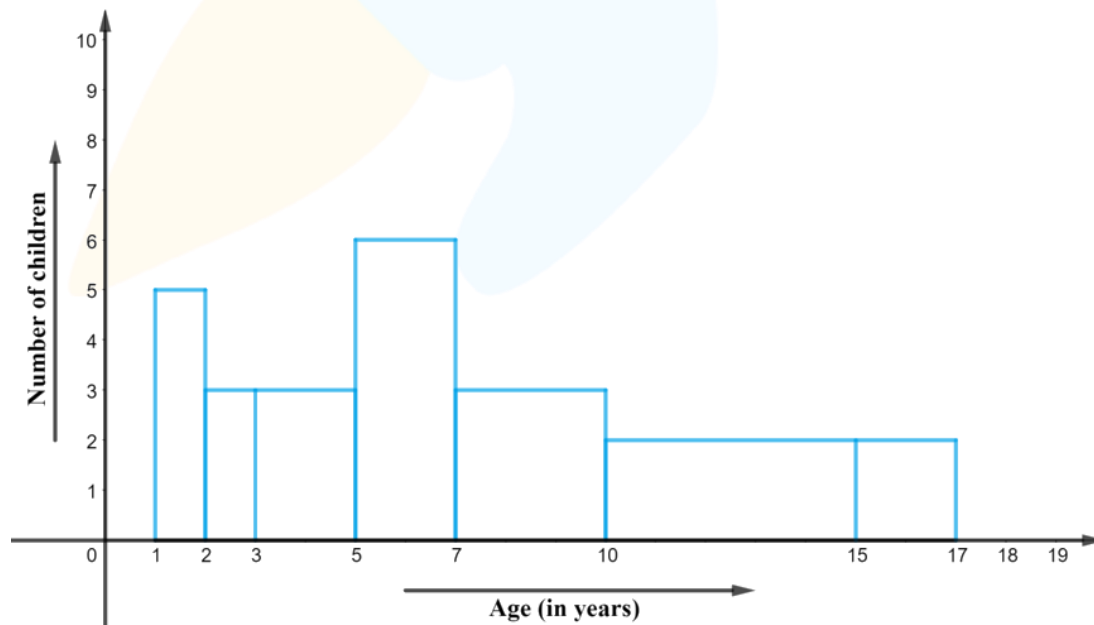
rectangle will be $\frac{10}{5} \times 1 = 2$.

We need to proceed in similar manner, to get the following table:

Age (in years)	Number of children	Width of the Class	Length of the rectangle
1 – 2	5	1	$\frac{5}{1} \times 1 = 5$
2 – 3	3	1	$\frac{3}{1} \times 1 = 3$
3 – 5	6	2	$\frac{6}{2} \times 1 = 3$
5 – 7	12	2	$\frac{12}{2} \times 1 = 6$
7 – 10	9	3	$\frac{9}{3} \times 1 = 3$
10 – 15	10	5	$\frac{10}{5} \times 1 = 2$
15 – 17	4	2	$\frac{4}{2} \times 1 = 2$

Since we have calculated these lengths for an interval of 1 year in each case, we may call these lengths as “proportion of children per 1-year interval”.

So, the histogram with varying width is as:



3. Frequency Polygons:

Frequency polygon is a visual way of representing quantitative data and its frequencies. Usually, we draw a frequency polygon with the help of the histogram by joining the **mid-points** of the upper sides of the adjacent rectangles of the histogram by means of line segments. When we join all by line segments, we obtain the polygon.

For example, the following table gives the length of 40 leaves of a plant, measured correct to one millimeter:

Length (in mm)	Number of Leaves
117.5 – 126.5	3
126.5 – 135.5	5
135.5 – 144.5	9
144.5 – 153.5	12
153.5 – 162.5	5
162.5 – 171.5	4
171.5 – 180.5	2

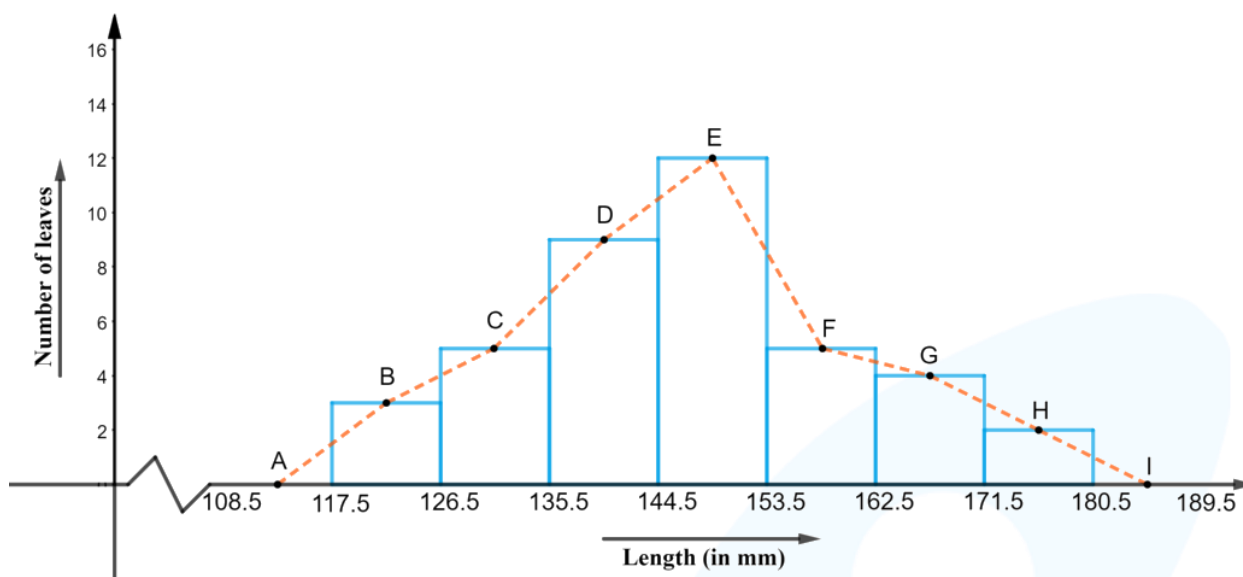
Let us represent the data given above graphically as follows:

First, we draw a Histogram graph of the given data then mark the mid-points of the upper sides of the adjacent rectangles of this histogram by means of line segments.

Now call these mid-points B, C, D, E, F, G and H. Joining the points by line segments, we obtain the figure BCDEFGH.

To complete the polygon, we assume that there is a class interval with frequency zero before 117.5 – 126.5 and one after 171.5 – 180.5, and their mid-points are A and I, respectively. ABCDEFGHI is the frequency polygon corresponding to the data given.

Hence, we obtain the frequency polygon graph as:



Frequency polygons can also be drawn independently without drawing histograms. For this, we require the mid-points of the class-intervals used in the data. These **mid-points** of the class-intervals are called **class-marks**.

To find the class-mark of a class interval, we find the sum of the upper limit and lower limit of a class and divide it by 2.

Thus,
$$\text{class-mark} = \frac{\text{upper limit} + \text{lower limit}}{2}.$$

Measures of Central Tendency:

We can make out some important features of the represented data by considering only certain representatives of the data by using measures of central tendency or averages.

1. Mean:

The mean or average of observations is the sum of the values of all the observations divided by the total number of observations.

$$\text{Mean} = \frac{\text{sum of the observations}}{\text{number of the observations}}$$

It is denoted by the symbol $\bar{x}$, read as 'x bar'.

If we are finding the mean for $x_1, x_2, x_3, \dots, x_n$ observations, writing $x_1 + x_2 + x_3 + \dots + x_n$ would be a tedious job.

We use the Greek symbol Σ (Sigma) for summation.

Instead of writing $x_1 + x_2 + x_3 + x_4 + \dots + x_n$, we write $\sum_{i=1}^n x_i$, which is read as 'the sum of x_i as i varies from 1 to n '.

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n} \quad \text{where } i \text{ varies from 1 to } n.$$

For example,

Example: The marks obtained by 10 students in a mathematics test as:

85 63 75 77 54 46 29 57 71 44

Find the mean of the marks.

Solution: Sum of the marks $= \sum_{i=1}^n x_i = 85 + 63 + 75 + 77 + 54 + 46 + 29 + 57 + 71 + 44 = 601$

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n} = \frac{601}{10} = 60.1$$

Therefore, Mean

Hence, mean of the marks is 60.1.

We know that if $x_1, x_2, x_3, \dots, x_n$ are observations with respective frequencies $f_1, f_2, f_3, \dots, f_n$ then this means, observation x_1 occurs f_1 times, x_2 occurs f_2 times, and so on.

Now, the sum of the values of all the observations $= f_1x_1 + f_2x_2 + \dots + f_nx_n$

and the number of observations $= f_1 + f_2 + \dots + f_n$

So, the mean of the data is given by $\bar{x} = \frac{f_1x_1 + f_2x_2 + \dots + f_nx_n}{f_1 + f_2 + \dots + f_n}$

or, $\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$ where i varies from 1 to n .

Let us take an example:

Example: A survey was conducted in which the number of children in 30 houses in a locality was recorded as:

0	1	2	2	1	2	3	1	3	2
4	3	1	4	2	2	4	1	2	1
3	1	1	1	1	2	3	2	2	3

Find the mean number of children per house.

Solution: Let us write the data in a tabular form:

Number of children (x_i)	Number of houses (f_i)	$f_i x_i$
0	1	0
1	10	10
2	10	20
3	6	18
4	3	12
TOTAL	30	60

$$\text{Mean } \bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{60}{30} = 2$$

Hence the mean number of children per house is 2.

2. Median:

The median is a measure of central tendency which gives the value of the **middle-most** observation in the data.

For finding the median of ungrouped data, we first arrange the data values of the observations in ascending order.

Then, if the number of observations, n is **odd**, the median is the $\left(\frac{n+1}{2}\right)^{th}$ observation. And, if n is **even**, then the median will be the average of the $\left(\frac{n}{2}\right)^{th}$ and the $\left(\frac{n}{2}+1\right)^{th}$ observations.

Let us consider an example for each case:

Example 1: The marks obtained by 10 students in a mathematics test as:

85 63 75 77 54 46 29 57 71

Find the median of the marks.

Solution: Let us arrange the marks in ascending order first

29 46 54 57 63 71 75 77 85

Here $n=9$, which is odd.

The median will be the $\left(\frac{n+1}{2}\right)^{th}$ observations.

i.e., the 5th observation which is 63.

Therefore, the median of the marks is 63.

Example 2: The marks obtained by 10 students in a mathematics test as:

85 63 75 77 54 46 29 57 71 44

Find the median of the marks.

Solution: Let us arrange the marks in ascending order first.

29 44 46 54 57 63 71 75 77 85

Here $n = 10$, which is even.

The median will be the average of the $\left(\frac{n}{2}\right)^{th}$ and the $\left(\frac{n}{2} + 1\right)^{th}$ observations.

i.e., the 5th and 6th observations.

5th observation is 57 and 6th observation is 63.

$$\text{So, Median} = \frac{57 + 63}{2} = \frac{120}{2} = 60$$

Therefore, the median of the marks is 60.

3. Mode:

A mode is that value among the observations which occurs most often, i.e., the value of the observation having the **maximum frequency**.

First, let us take an example to find the mode of ungrouped data:

Example: A survey was conducted in which the number of children in 30 houses in a locality was recorded as:

0	1	2	2	1	2	3	1	3	0
1	3	1	1	2	2	0	1	2	1
3	0	0	1	1	2	3	2	2	0

Find the mode number of children per house.

Solution: Let us write the data in a tabular form:

Number of children	Number of houses
0	6
1	10
2	9
3	5
TOTAL	30

From the table, it can be observed that the maximum frequency is 10, belonging to a family having only 1 child. Therefore, the mode number of children per house is 1.

PROBABILITY

The words ‘probably’, ‘doubt’, ‘most probably’, ‘chances’, etc., involve an element of uncertainty. The uncertainty of ‘probably’ etc. can be measured numerically by means of ‘probability’ in many cases.

Trial:

A trial is an action which results in one or several outcomes.

Event:

An event for an experiment is the collection of some outcomes of the experiment.

Experimental or empirical probabilities:

Experimental probabilities are based on the results of actual experiments and adequate recordings of the events occurring.

Suppose a coin was tossed 100 times and the frequencies of the outcomes were as follows:

Head: 45 and Tail: 55

Based on this experiment, the empirical probability of a head is $\frac{45}{100}$ or 0.45 and that of getting a tail is $\frac{55}{100}$ or 0.55.

These probabilities are based on the results of an actual experiment of tossing a coin 100 times. For this reason, they are called experimental or empirical probabilities.

These probabilities are only ‘**estimates**’ and if we perform the same experiment for another 1000 times, we may get different data giving different probability estimates.

Let n be the total number of trials. The empirical or experimental probability $P(E)$ of an event E occurring, is given by

$$P(E) = \frac{\text{Number of trials in which the event happened}}{\text{The total number of trials}}$$

NOTE:

The probability of an event E is a number $P(E)$ such that, $0 \leq P(E) \leq 1$. i.e., the probability of an event lies between 0 and 1, where 0 and 1 are inclusive.

Impossible event:

The probability of an event which is impossible to occur is **0**. Such an event is called an impossible event.

Sure event:

The probability of an event which is sure or certain to occur is **1**. Such an event is called a sure event or a **certain event**.

Let us consider some examples.

Example: 1500 families with 2 children were selected randomly, and the following data were recorded:

Number of girls in a family	2	1	0
Number of families	475	814	211

Compute the probability of a family, chosen at random, having

- (i) 2 girls (ii) 1 girl (iii) No girl

Also, check whether the sum of these probabilities is 1.

Solution:

Total number of families = 1500

Number of families having 2 girls = 475

Number of families having 1 girl = 814

Number of families having no girl = 211

(i) Probability of family having 2 girls, $P_1 = \frac{\text{Family having 2 girls}}{\text{Total number of families}}$

Therefore, $P_1 = \frac{475}{1500} = \frac{19}{60}$

(ii) Probability of family having 1 girl, $P_2 = \frac{\text{Family having 1 girl}}{\text{Total number of families}}$

Therefore, $P_2 = \frac{814}{1500} = \frac{407}{750}$

(iii) Probability of family having no girl, $P_3 = \frac{\text{Family having no girl}}{\text{Total number of families}}$

Therefore, $P_3 = \frac{211}{1500}$

Sum of all the three probabilities = $P_1 + P_2 + P_3$

$$= \frac{19}{60} + \frac{407}{750} + \frac{211}{1500} = \frac{475 + 814 + 211}{1500} = \frac{1500}{1500} = 1$$

Example: A die is thrown once. Find the probability of getting

(i) a prime number, (ii) a number lying between 2 and 6, (iii) an odd number.

Solution:

All possible outcomes when we throw a die are 1, 2, 3, 4, 5 and 6.

Therefore, total number of outcomes, $n = 6$

(i) Prime numbers on dice are 1, 3 and 5

Number of prime numbers, $a = 3$

$$\text{Probability (getting a prime number)} = \frac{a}{n} = \frac{3}{6} = \frac{1}{2}$$

(ii) Numbers lying between 2 and 6 are 3, 4 and 5

Number of numbers lying between 2 and 6, $b = 3$

$$\text{Probability (getting a number lying between 2 and 6)} = \frac{b}{n} = \frac{3}{6} = \frac{1}{2}$$

(iii) Odd numbers on die are 1, 3 and 5

Number of odd numbers, $c = 3$

$$\text{Probability (getting an odd number)} = \frac{c}{n} = \frac{3}{6} = \frac{1}{2}$$