Vectors And 3-D Geometry

ASSIGNMENT - QUESTIONS

LEVEL - II

Q. 19 In ΔABC, prove that $\cos 2A + \cos 2B + \cos 2C \geq \frac{-3}{2}$

Q. 20 The internal bisectors of the angles A, B and C of ΔABC meet the opposite sides in D, E and F respectively. Prove that

$$\text{area } (\Delta DEF) \leq \frac{1}{4} \text{area } (\Delta ABC)$$

Q. 21 Let $\vec{A}(t) = f_1(t)\hat{i} + f_2(t)\hat{j}$ and $\vec{B}(t) = g_1(t)\hat{i} + g_2(t)\hat{j}$, $t \in [0,1]$ where f_1, f_2, g_1, g_2 are continuous functions.

If $\vec{A}(t)$ and $\vec{B}(t)$ are non-zero vectors for all t and $\vec{A}(0) = 2\hat{i} + 3\hat{j}$, $\vec{A}(1) = 6\hat{i} + 2\hat{j}$, $\vec{B}(0) = 3\hat{i} + 2\hat{j}$ and $\vec{B}(1) = 2\hat{i} + 6\hat{j}$, then show that $\vec{A}(t)$ and $\vec{B}(t)$ are parallel for some t.

Q. 22 **(a)** Prove that the segment joining the middle points of the two non-parallel sides of a trapezium is parallel to the parallel sides and equal to half their sum.

(b) Prove that the segment joining the mid-points of the diagonals of a trapezium is parallel to the parallel sides and equal to half their difference.

Q. 23 In a triangle ABC, D, E, F are taken on BC, CA and AB respectively such that $\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = n$.

Prove that $\text{area } (\Delta DEF) = \frac{n^2 - n + 1}{(n+1)^2} \text{area } (\Delta ABC)$

Q. 24 Let $\vec{a}, \vec{b}, \vec{c}$ be three non-coplanar vectors, so that $[\vec{a} \;\; \vec{b} \;\; \vec{c}] \neq 0$. Define the set of vectors, $\vec{a}_1, \vec{b}_1, \vec{c}_1$ as

$$\vec{a}_1 = \frac{\vec{b} \times \vec{c}}{[\vec{a} \;\; \vec{b} \;\; \vec{c}]}, \quad \vec{b}_1 = \frac{\vec{c} \times \vec{a}}{[\vec{a} \;\; \vec{b} \;\; \vec{c}]}, \quad \vec{c}_1 = \frac{\vec{a} \times \vec{b}}{[\vec{a} \;\; \vec{b} \;\; \vec{c}]}$$

Observe that

$$\vec{a} \cdot \vec{a}_1 = \vec{b} \cdot \vec{b}_1 = \vec{c} \cdot \vec{c}_1 = 1$$

The system of vectors $\vec{a}_1, \vec{b}_1, \vec{c}_1$ is called the **reciprocal system** of the set of vectors $\vec{a}, \vec{b}, \vec{c}$.

(a) Prove that $\left[\vec{a}_1 \ \vec{b}_1 \ \vec{c}_1\right] = \dfrac{1}{\left[\vec{a} \ \vec{b} \ \vec{c}\right]}$

(b) Prove that $\vec{b}_1 \times \vec{c}_1 + \vec{c}_1 \times \vec{a}_1 + \vec{a}_1 \times \vec{b}_1 = \dfrac{\vec{a} + \vec{b} + \vec{c}}{\left[\vec{a} \ \vec{b} \ \vec{c}\right]}$

(c) Find explicitly the reciprocal system of the set of vectors $\vec{a} = \hat{i} + \hat{j} - \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} + 2\hat{j} - \hat{k}$ and prove that it satisfies the two properties above.

Q. 25 Points P, Q, R divide BC, CA and AB of ΔABC in the ratio 1 : 2. The segments AP, BQ and CR form the triangle XYZ. Prove that ΔABC and ΔXYZ have the same centroid.

Q. 26 Let $\vec{u}$ and $\vec{v}$ be two given non-collinear unit vectors and $\vec{w}$ be a vector such that $\vec{w} + \left(\vec{w} \times \vec{u}\right) = \vec{v}$.

Prove that $\left|\left(\vec{u} \times \vec{v}\right) \cdot \vec{w}\right| \leq \dfrac{1}{2}$

Q. 27 Three concurrent straight lines OA, OB, OC are produced to D, E, F respectively. AB and DE, BC and EF, CA and FD intersect in X, Y, Z respectively. Prove that X, Y, Z are collinear.

Q. 28 Let $OABC$ be a regular tetrahedron of side L. D is the circumcentre of ΔOAB and E is the mid-point of AC. Find DE.

Q. 29 Prove that the point of intersection of the diagonals of a trapezium lies on the line passing through the mid-points of the parallel sides.

Q. 30 Show that the angle between any edge and a face not containing that edge of a rectangular tetrahedron is $\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$.