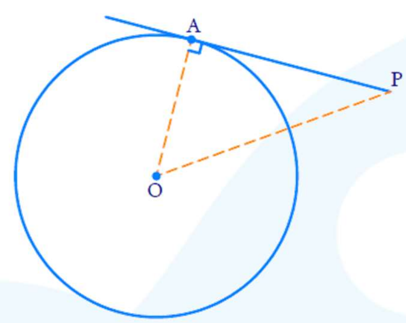
Circles and Tangents

Review Exercise Answers

Level-1

Single Choice

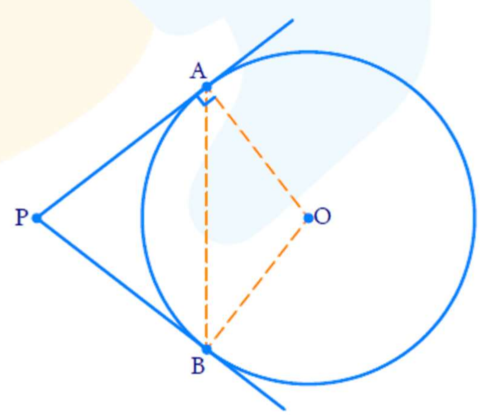
S1. (B). Consider the following figure:



The radius equals

$$OA = \sqrt{OP^2 - PA^2} = \sqrt{10^2 - 8^2} = 6 \text{ cm}$$

S2. (A). Consider the following figure:



Since PA = PB, we note that $\angle PAB = \angle PBA$. Now, in ΔPAB, we have:

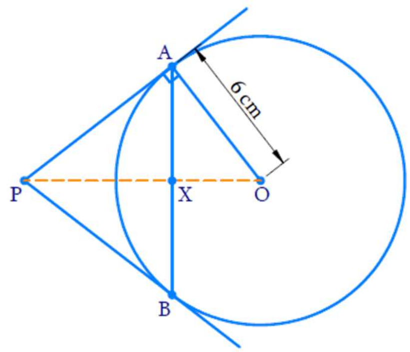
$$\angle APB + \angle PAB + \angle PBA = 180^\circ$$

$$\Rightarrow \angle \text{APB} + 2(\angle PAB) = 180^\circ$$

$$\Rightarrow \angle APB + 2(90^\circ - \angle OAB) = 180^\circ$$

$$\Rightarrow \angle APB = 2\angle OAB$$

S3. (B). Observe the following figure:



Note that $AX = \frac{1}{2}AB = 4.8$ cm. Thus,

$$OX = \sqrt{OA^2 - AX^2} = \sqrt{6^2 - 48^2} = 3.6 \text{ cm}$$

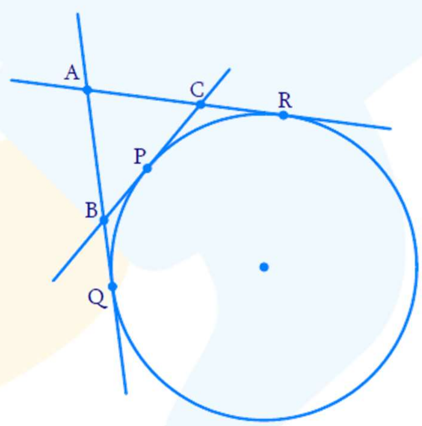
Now, using the Pythagoras Theorem, we have:

$$PX = \sqrt{PA^2 - AX^2}, \text{ and}$$

$$PX + OX = \sqrt{PA^2 + AO^2}$$

Eliminating PX from these equations gives PA = 8 cm (verify).

S4. (C). Consider the following figure:

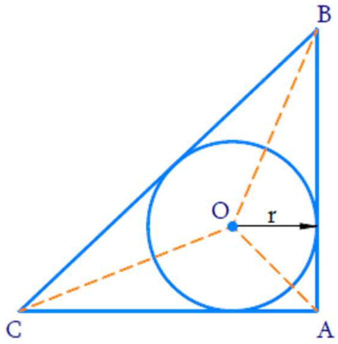


We have:

$$\begin{aligned} \ell &= AB + BC + AC \\ &= (AB + BP) + (CP + AC) \\ &= (AB + BQ) + (AC + CR) \\ &= AQ + AR \\ &= AQ + AQ \\ &= 2AQ \end{aligned}$$

We have used the fact that the two tangents drawn from an external point to a circle are of equal lengths.

S5. (B). Observe the following figure:



Note that

$$BC = \sqrt{AB^2 + AC^2} = 10 \text{ cm}$$

Now, the area of $\angle AOB$ will be

$$\Delta_1 = \frac{1}{2} \times AB \times r$$

Similarly the areas of ΔBOC and ΔCOA will be

$$\Delta_2 = \frac{1}{2} \times BC \times r, \Delta_3 = \frac{1}{2} \times CA \times r$$

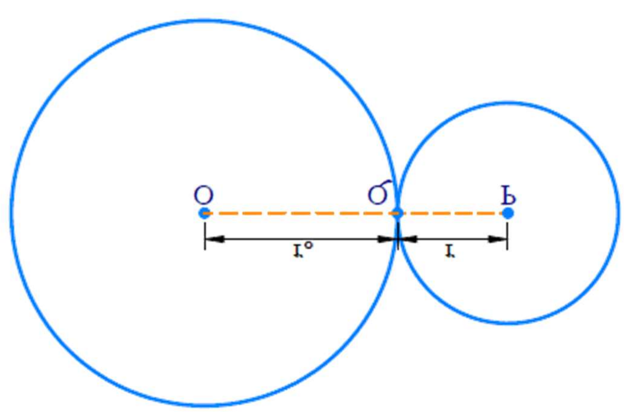
Adding the three areas should give us the area of ΔABC, which is 24 cm^2:

$$\frac{1}{2}(AB + BC + CA)r = 24 \text{ cm}^2$$

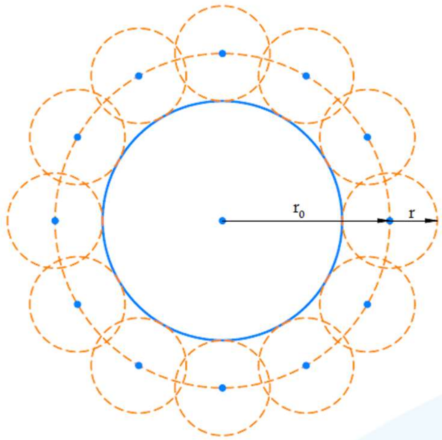
$$\Rightarrow \quad r = 2 \text{ cm}$$

Miscellaneous

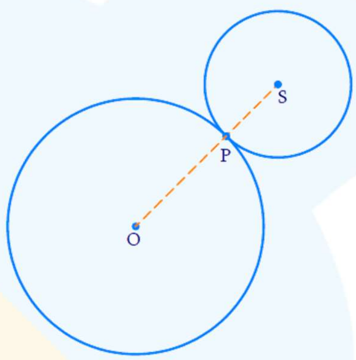
S6. Let the center of the fixed circle be O and let its radius be r_0. Let the (fixed) radius of the variable circle be r. In the following figure, we have shown one of the variable circles with center P touching the fixed circle at Q:



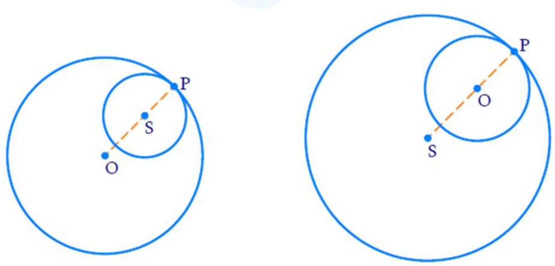
Since OQP is a straight line (why?), OP = OQ + QP or OP = r_0 + r.
Thus, OP is fixed, that is, the distance of P from O is constant. Clearly, the locus of the centers is a circle with center O and radius r_0 + r. This is shown explicitly in the following figure:



S7. Let the fixed circle have the center O, and let the fixed point of contact be P. The following figure shows one of the variable circle with center S touching the fixed circle at P:

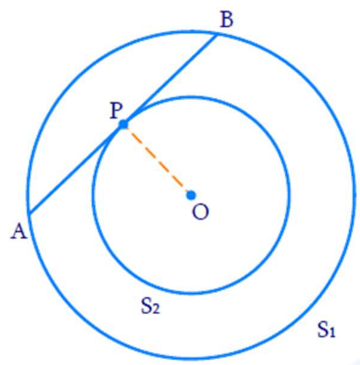


Clearly, S (the center of the variable circle) will be collinear with O and P, no matter what variable circle we take. The variable circle can even touch the fixed circle internally, or get touched by the fixed circle internally:



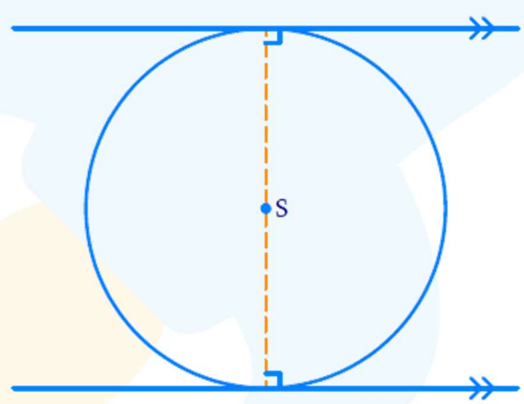
In all cases, S is collinear with O and P. Thus, the locus of S (the center of the variable circle) is the straight line passing through O and P.

S8. Consider the following figure. Since AB is a tangent for S_2, the radius OP of S_2 must be perpendicular to AB:

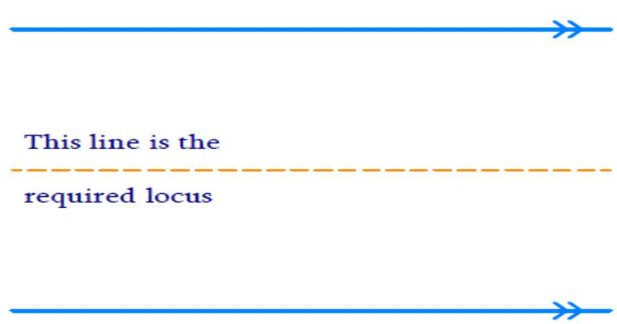


Now, in S_1, OP is the perpendicular dropped from the center O to the chord AB. Clearly, P must be mid-point of AB, so AP = PB.

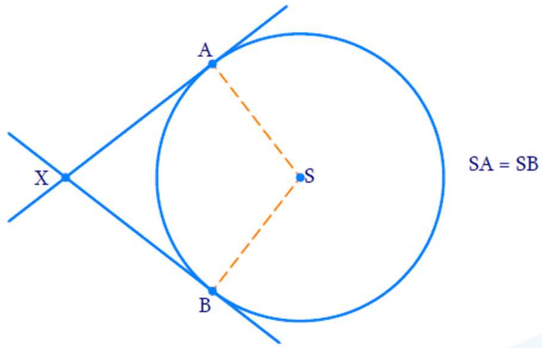
S9. (a) Consider the following figure:



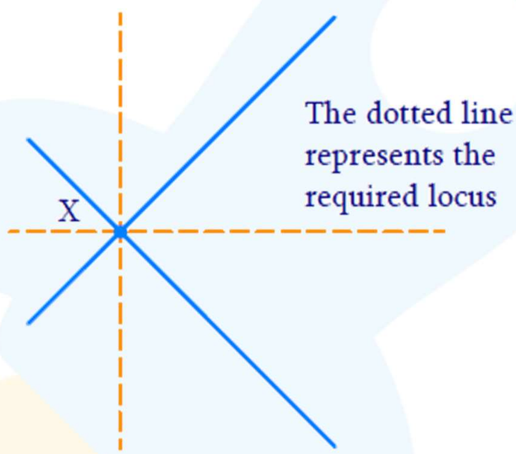
The center S of the circle is equidistant from the two parallel lines. Thus, the locus of S is the line parallel to the two given lines and placed midway between the two lines:



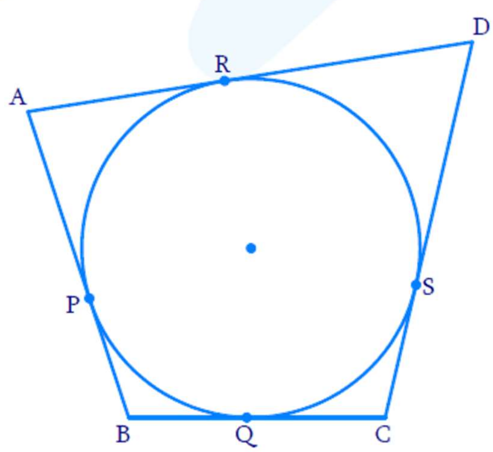
(b) In this case also, the center S of the (variable) circle should be equidistant from the two fixed (intersecting) lines:



The, S must lie on one of the two angle bisector of the angle[s] formed at X:



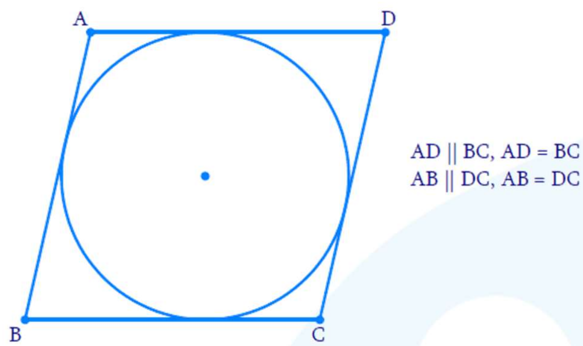
S10. (a) Consider the following figure:



We have:

$$
\begin{aligned}
AB + CD &= (AP + BP) + (CR + DR) \\
&= (AS + BQ) + (CQ + DS) \\
&= (AS + DS) + (BQ + CQ) \\
&= AD + BC
\end{aligned}
$$

(b) Consider the following figure, in which ABCD is parallelogram and a circle is inscribed inside it.



Using the result of part-(a), we have:

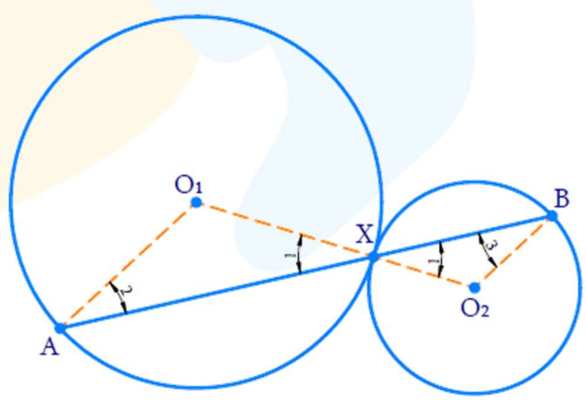
$$AB + DC = AD + BC$$

$$\Rightarrow \quad AB + AB = AD + AD$$

$$\Rightarrow \quad AB + AD$$

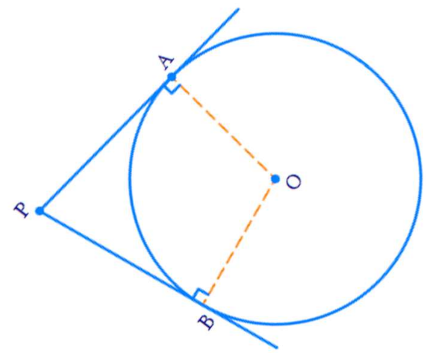
Clearly, ABCD is a rhombus.

S11. Consider the following figure. Note that O_1, X and O_2 are collinear:



Clearly, $\Delta_1 AX$ and $\Delta O_2 BY$ are both isosceles. Thus, $\angle 1 = \angle 2$ in $\Delta O_1 AX$ and $\angle 1 = \angle 3$ in $\Delta O_2 BX$. This means that $\angle 2 = \angle 3$, from which we conclude that $O_1 A \parallel O_2 B$.

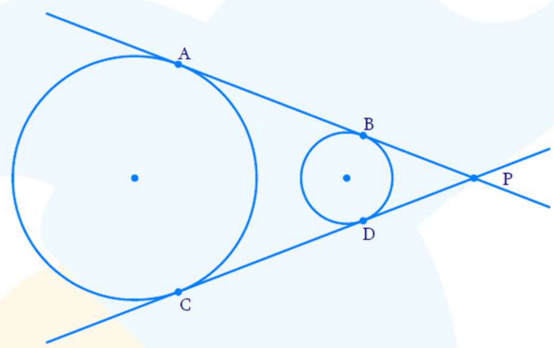
S12. Consider the following figure. Note that $\angle PAO = \angle PBO = 90^o$, since tangent is perpendicular to the radius through the point of contact:



Also, in quadrilateral PAOB, the sum of the all the angles must be 360º. Thus,

$$\angle APB + \angle AOB = 360^o - (\angle PAO + \angle PBO)$$
$$= 360^o - 180^o$$
$$= 180^o$$

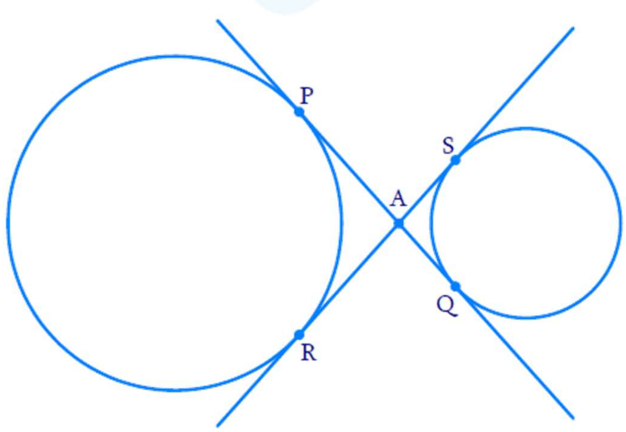
S13. (a) Consider the following figure:



Note that PA = PC and PB = PD. Thus, PA – PB = PC – PD

$\Rightarrow$ = AB = CD

(b) Consider the following figure:

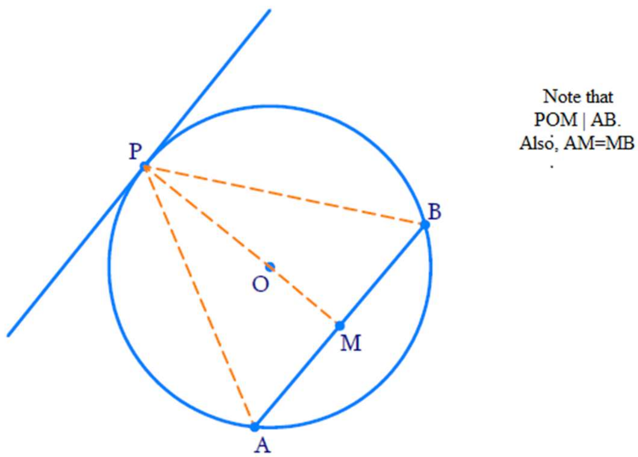


Note that AP = AR and AQ = AS, and so:

AP + AQ = AR + AS

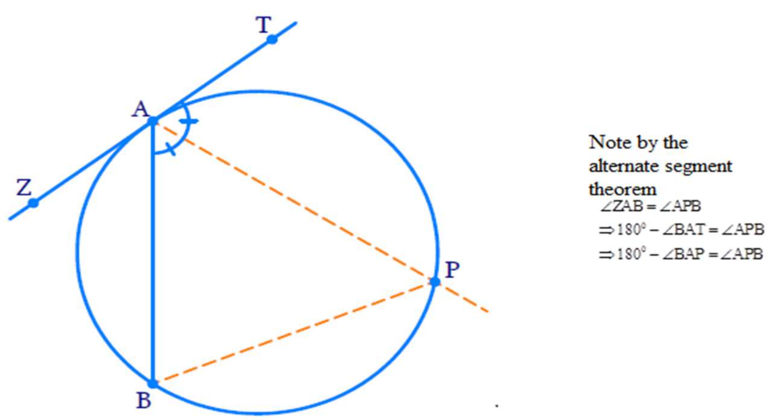
$\Rightarrow$ PQ = RS

S14. Consider the following figure:



From the observations made in the figure, POM is the perpendicular bisector of AB. Thus, AP = PB, which means that $\overset{\frown}{AP} = \overset{\frown}{PB}$, that is, P bisects the are $\overset{\frown}{APB}$.

S15. In the following figure AT is the tangent to the circle at A, while AP is the angle bisector of $\angle BAT$. We have joined BP:



Using the observations made in the figure, we have:

$180^o - 2\angle BAP = \angle APB$

$= 180^o - (\angle BAP + \angle ABP)$

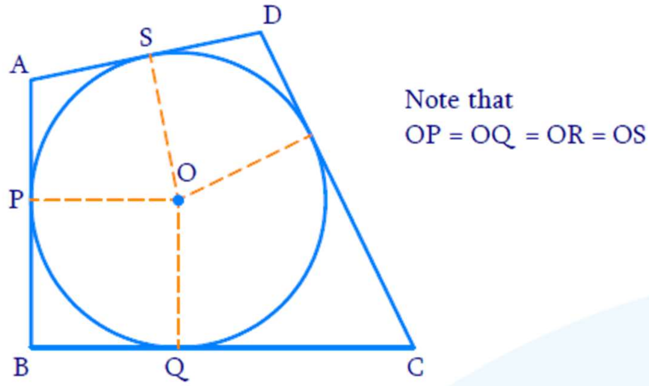
$\Rightarrow\ 2\angle BAP = \angle BAP + \angle ABP$

$\Rightarrow\ \angle BAP = \angle ABP$

$\Rightarrow\ PA = PB$

Thus, $\overset{\frown}{AP} = \overset{\frown}{BP}$, which means that P is the mid-point of $\overset{\frown}{APB}$.

S16. Consider the following figure, in which O is the center of the circle inscribed in quadrilateral ABCD:



Clear, O is equidistant from all the four sides of the quadrilateral. Now, since OS = OP, O lies on the angle bisector of $\angle A$. Similarly, since OP = OQ, O lies on the angle bisector of $\angle B$. Continuing this way, we see that O lies on the angle bisectors of $\angle C$ and $\angle D$ also. Thus, the bisector of the angles of the quadrilateral are concurrent, the point of concurrent being O.