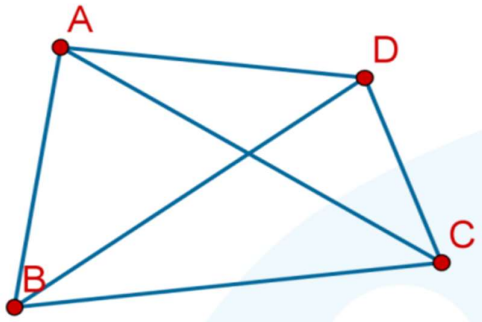
Triangles and Quadrilaterals

Review Exercise Answers

Level-2

Single Choice Correct Only

S1. (D). Carefully observe the following figure (which is not accurate, as we still have not determined the properties of ABCD):



Considering the sum of the distances of vertices A and C from the others, we have:

AB + AC + AD = CB + CD + CA
➔ AB + AD = BC + CD.

Similarly, we can say that

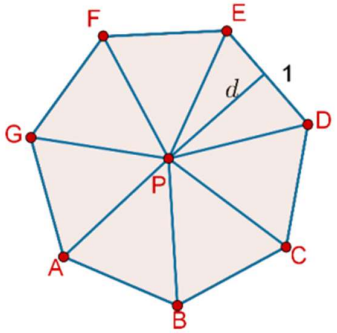
AB + BC = AD + CD.

Adding the last two relations, it is easy to conclude that AB = CD. Similarly, AD = BC. Hence, ABCD is definitely a parallelogram. Now, considering the sum of the distances of vertices A and D from the others, we have:

AB + AC + AD = DA + DB + DC

Since AD = DC, this implies that AC = BD. Thus, ABCD is a parallelogram in which the diagonals are equal. This means that ABCD is a rectangle.

S2. (A). The following figure shows (as an example) a 7-sided polygon, and a point P inside it:



P has been joined to each of the vertices, and the distance of P from the side DE has been marked *d*. Let the length of each side of the polygon be l. We have:

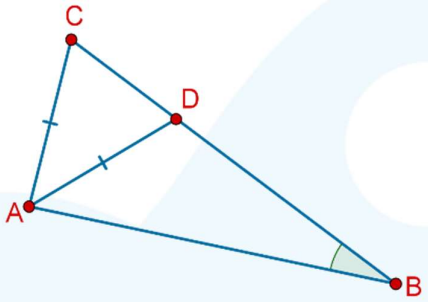
Area(ΔPDE) = ½ × DE × d
➔ d = 2 × Area(ΔPDE)/DE
= 2 × Area(ΔPDE)

Such an equation can be written for each of the sub-triangles formed within the polygon. As P moves, the areas of these triangles will vary. However, *their sum will be a constant*, equal to the area of the polygon. Using this fact and the relation we have written above, it is easy to show that the sum of the distances of P from each side must be a constant. Completing the proof is left to you as an exercise.

Multiple Options May be Correct

S3. (A) and (C). Let us discuss each of the first four options one-by-one.

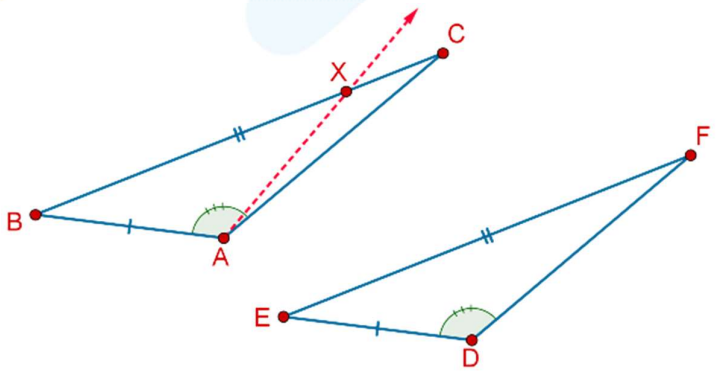
Option (A): If the non-included equal angles are acute, then the two triangles may not be congruent, as the following figure shows:



In this figure, compare ΔDAB and ΔCAB. Note that DA = CA, and AB = AB (common). Thus, two pairs of sides are equal. Also, ∠B = ∠B (common), which means that a pair of non-included angles are equal. Even then, ΔDAB is clearly not congruent to ΔCAB. Therefore, this option is correct.

Option (B): If the non-included equal angles are right angles, then the RHS criterion applies, so the two triangles must be congruent (make a diagram to confirm this). Therefore, this option is incorrect.

Option (C): If the non-included pair of angles have obtuse values, we have the following situation:



In this case, the two triangles *must be congruent*. Can you see why? Try to prove this by contradiction. Suppose that ∠A is not equal to ∠D. Let ∠A be larger. Then, draw a ray through A inclined to AB at an angle equal to ∠D, and intersecting BC at X. Compare

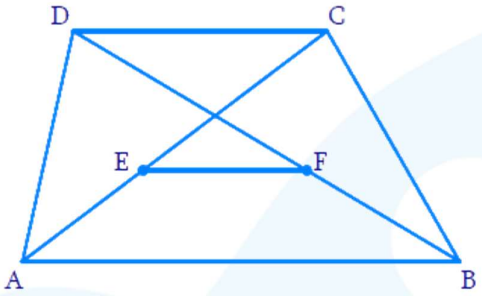
ΔBAX with ΔEDY, and see what happens. You will arrive at a contradiction which will prove that $\angle A$ must be equal to $\angle D$, and hence ΔABC and ΔDEF must be congruent.

Option (D): Clearly, this option is incorrect, as the first figure above shows. It is not necessary for the included angles to be equal.

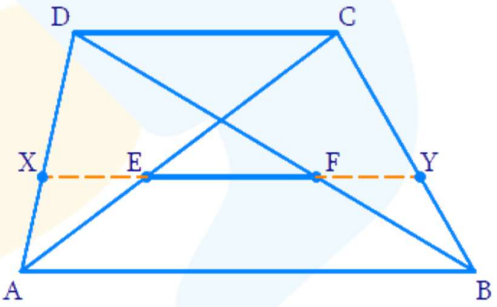
Integer Answers

Miscellaneous

S4. (a) Consider the following figure, in which ABCD is a trapezium with AB || CD:



E is the mid-point of AC while F is the mid-point of BD. We need to show that EF || AB || CD, and in the second part, we need to evaluate the magnitude of EF. Let X and Y be the mid-points of AD and BC respectively. Join EX and BY:



First, we will prove that X, E, F and Y are collinear. This may be *visually* obvious but we need to demonstrate it rigorously. Using the mid-point theorem, we have:

- In ΔACD, $XE \parallel DC$.
- In ΔADB, $XF \parallel AB$. But $AB \parallel DC$, which means that $XF \parallel DC$.

Thus, both XE and XF are parallel to DC, i.e., X, E and F must be collinear. Similarly, Y, E and F are collinear.

Since X, E and F are collinear and XF || AB || DC, we have:

EF || AB || DC

(b) Using the mid-point theorem, we have:

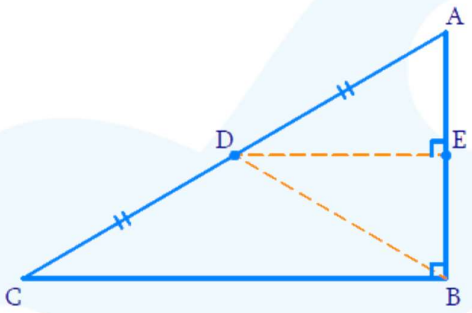
- In ΔADC, $XE = \frac{1}{2}DC$
- ΔADB, $XE = \frac{1}{2}AB$

Subtracting the two relations, we have:

$$XF - XE = \frac{1}{2}(AB - DC)$$

$$\Rightarrow \quad EF = \frac{1}{2}(AB - DC)$$

S5. You may want to solve this problem using trigonometry, if you are familiar with it. However, we will use a "pure" geometrical approach consider the following figure:

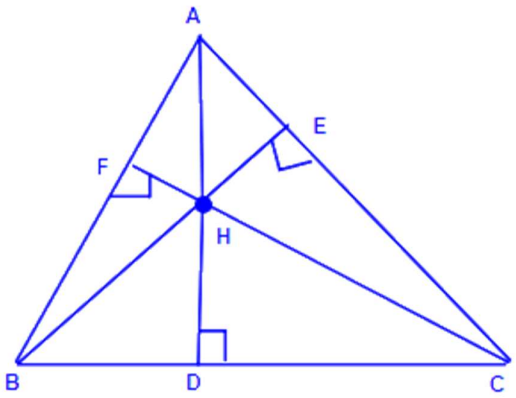


AD = CD = AB

D is the mid-point of AC and we have joined BD. From D, we have dropped $DE \perp AB$. This means that $DE \| CB$. And since D is the mid-point of AC, we can conclude using the mid-point theorem that E must be the mid-point of AB. Now, compare ΔAED and ΔBEO .Since $AE = EB$, $\angle AED = \angle BED = 90^\circ$ and DE is common, the two triangles are congruent by the SAS criterion. This means that AD = BD. Also, since AD = AB, we have AD = BD = AD.

Thus, ΔABD is equilateral, which means that $\angle A = 60^\circ$, and thus $\angle C = 90^\circ - \angle A = 30^\circ$. Clearly, $\angle A = 2\angle C$.

S6. Recall that the orthocenter of a triangle is the point at which the three altitudes of the triangle meet. The orthocenter may lie inside or outside the triangle. Now, consider the following figure:



AD, BE and CF are the three altitudes of ΔABC which are concurrent at H. We now want to determine the orthocenter of ΔBHC. For that, let us find out the three altitudes of ΔBHC :

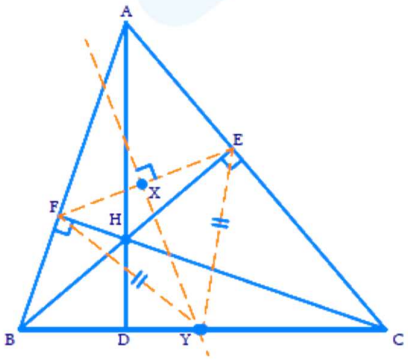
- The altitudes from H to BC is the line (segment) HD.
- What is the altitude from B to HC? Note that BF is perpendicular to HC (extended). Thus, the altitude from B to HC is BF.
- What is the altitude from C to HB? Once again, CE is perpendicular to HC (extended), and thus the altitude from C to BH is CE.

Where do the three altitudes of ΔBHC , namely HD, BF and CE meet? When extended, the three meet at the point A! Thus, A is the orthocenter of ΔHBC .

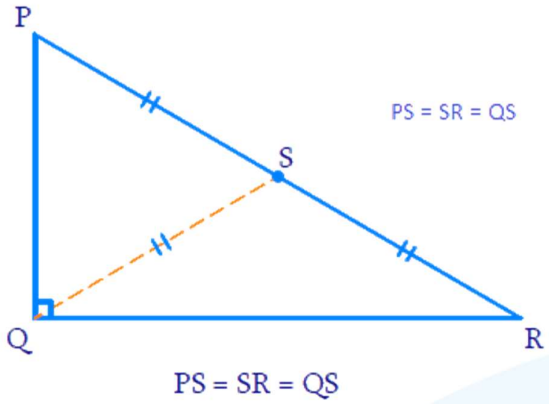
S7. We will present two ways to prove this result. The second method makes use of results from circles; if you cannot understand it right now, you can come back to it later after you've covered the chapter on circles.

Method1

Consider the following figure:



We have to prove that Y is the midpoint of BC. We will make use of the result that if we join the right-angled vertex of a right triangle to the mid-point of the hypotenuse, we obtain three equal segments, as shown below:



Coming back to our original proposal, suppose that Y is not mid-point of BC. Let Z be the mid-point of BC, as shown in the figure below. Join EZ and FZ.

Since BEC and BFC are right triangles and Z is the (assumed) mid-point of BC, we have

$EZ = BZ = CZ$

$FZ = BZ = CZ$

$\Rightarrow \quad EZ = FZ$

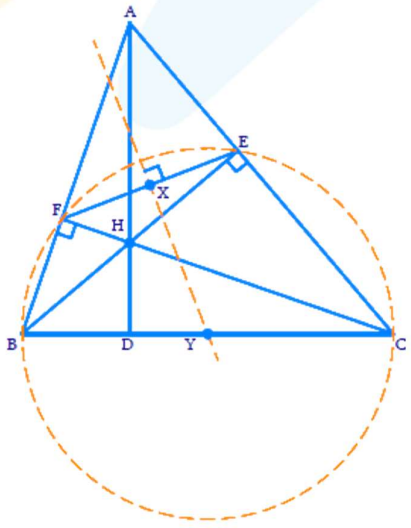
This should mean that Z lies on the perpendicular bisector of EF (why?), which is clearly not the case. Thus, the our assumption that Z (a point different from Y) is the mid-point is incorrect. The mid-point of BC is Y only.

Method2

Draw a circle with BC as diameter. It must pass through the points E and F, since

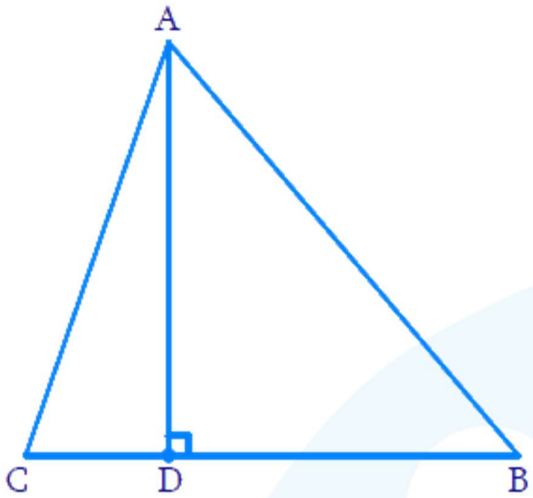
$\angle BEC = \angle BFC = 90^{\circ}$,

and angles in semi-circles are right angles:

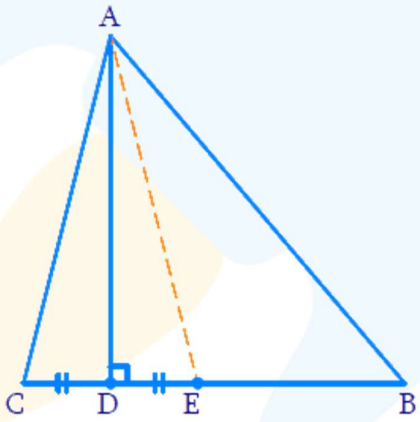


Clearly, EF is a chord of this circle. The perpendicular bisector of any chord passes through the circle's center, and bisects any diameter. Since BC is a diameter of the circle, the perpendicular bisector of EF will bisect BC. Very elegant!

S8. Consider the following figure, which shows the specific case of $BD > DC$:



We can make use of the Pythagoras Theorem in our proof (how?), but let's adopt a different approach. Through A, draw AE with E lying on DB such that CD = DE.

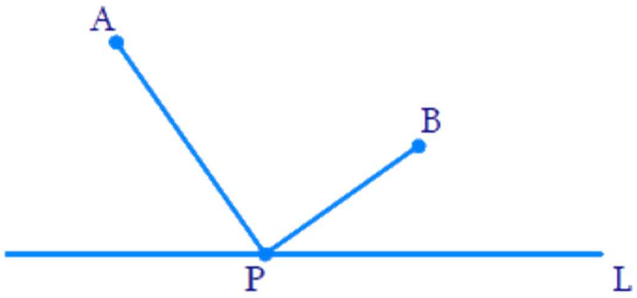


We make the following observations:

- $\Delta ACD \cong \Delta AED$ by the SAS criterion, which means that AC = AE.
- In ΔAEB, $\angle AEB$ must be obtuse (why?). This means that $\angle AEB > \angle ABE$, so that $AB > AE$.

From these two observations, we conclude that AB > AC. In a similar manner, assertion (c) can be proved. The proof of assertion (b) should be obvious.

S9. Consider the following figure.

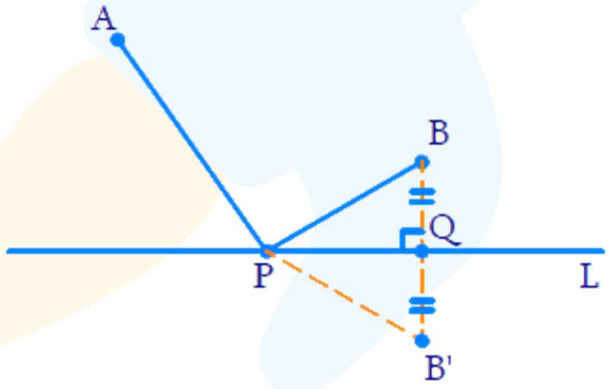


Clearly, as P moves on the line, the sum

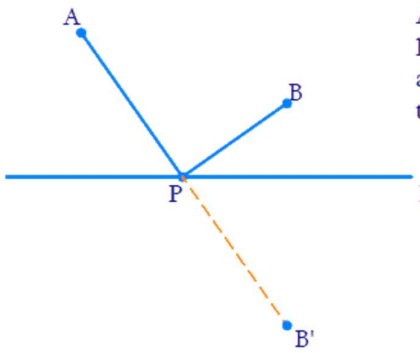
AP + PB will be minimum. Can you understand how? Suppose that initially, P is to the extreme left on the line. Both AP and PB will have very large values and their sum will be very large. As P is moved to the right, the value of this sum will decrease. However, as P is moved even further to the right, AP + PB will start increasing again. Try to visualize this variation mentally.

To determine the location of P for which

AP + PB is minimum, we will adopt a very ingenious technique. Take the ***mirror reflection*** of the point B in the line L, to obtain the point B'. Join P to B':

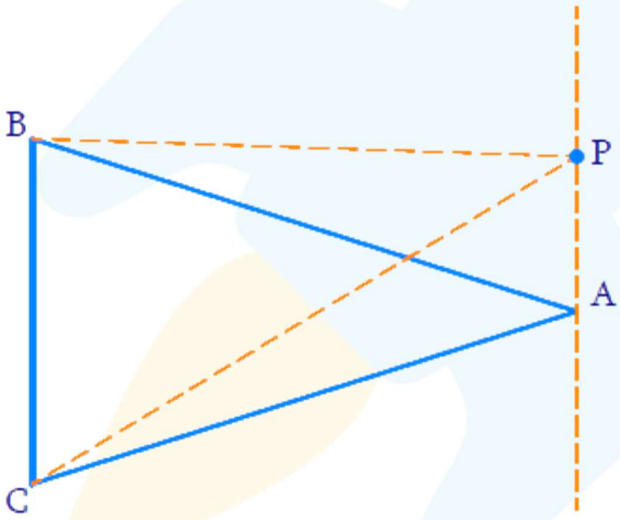


Note that $\angle PBQ \cong \Delta PB'Q$ by the SAS criterion, and so, $PB = PB'$. Thus, AP + PB is the same as $AP + PB'$. This means that determining the minimum of AP + PB is the same as determining the minimum of $AP + PB'$. When will $AP + PB'$ be minimum? Obviously, when the location of P is such that APB' forms a straight line:

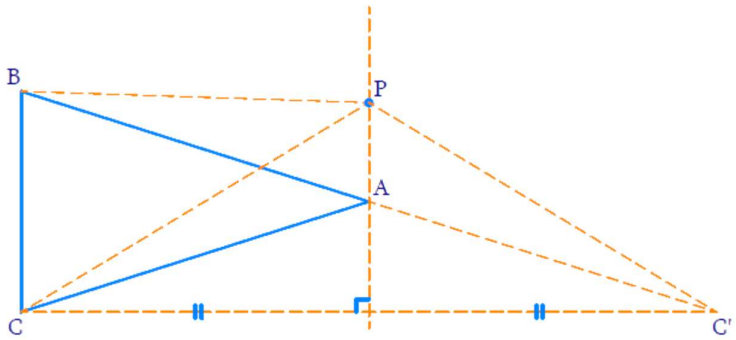


As an exercise, suppose that the distance of A and B from the line L are specified. How will you compute the numerical value of the location of P?

S10. Consider the following figure:



We have to show that as long as P is a point different from A on the line L, PB + PC will always be greater than AB + AC.
We take the mirror reflection of the point C in the line L, to obtain the point C'. We joint PC' and AC':



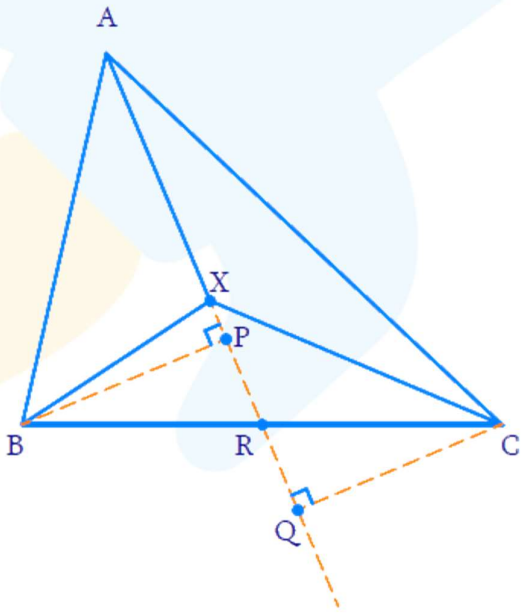
Note that $AC = AC'$ and $PC = PC'$, and thus:

$$AB + AC = AB + AC'$$

$$PB + PC = PB + PC'$$

Also, note that the points B, A and C' are collinear, due to the fact that ΔABC is isosceles (how do we show co linearity?). Thus, since BAC' is a straight line, $BA + AC'$ must be smaller than $BP + PC'$, and so: $PB + PC > AB + AC$.

S11. If you have any confusion about the concept of locus, you are urged to review the relevant discussion once again. In the following figure, X is a point such that the areas of ΔAXB and ΔAXC are equal:



Note how we have extended AX and dropped perpendiculars BP and CQ onto it. Now,

$$\text{area}\left(\Delta AXB\right) = \frac{1}{2} \times AX \times BP$$

$$\text{area}\left(\Delta AXC\right) = \frac{1}{2} \times AX \times CQ$$

If these two areas are equal, this must mean that BP = CQ. Thus, X must be positioned or located such that the perpendiculars dropped from B and C onto AX are equal in length.

This means that $\Delta BRP \cong \Delta CRQ$ by the ASA criterion, so that BR = CR, i.e., R is the mid-point of BC or AR is the median.

Thus, X can lie anywhere on the median (or median extended). For all such positions of X, the areas of ΔAXB and ΔAXC will be equal.