Numbers and Number Systems

Review Exercise Answers

Level-2

Single Choice Correct Only

S1. The correct option is (C). We have:

$$S=\frac{1}{1+\sqrt{2}+\sqrt{3}}\times\frac{1+\sqrt{2}-\sqrt{3}}{1+\sqrt{2}-\sqrt{3}}$$
$$=\frac{1+\sqrt{2}-\sqrt{3}}{\left(1+\sqrt{2}\right)^2-3}=\frac{1+\sqrt{2}-\sqrt{3}}{2\sqrt{2}}$$
$$=\frac{1+\sqrt{2}-\sqrt{3}}{2\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{2+\sqrt{2}-\sqrt{6}}{4}$$

S2. (A). Let $a_n=\sqrt{n-1}+\sqrt{n+1}$ be rational. Then $\frac{1}{a_n}$ is also rational. We have:

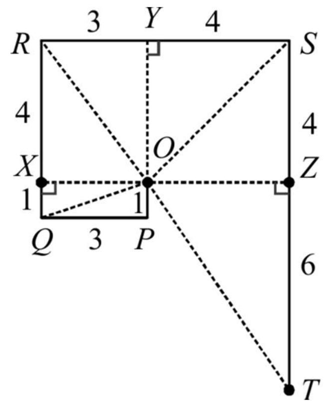
$$\frac{1}{a_n}=\frac{1}{\sqrt{n-1}+\sqrt{n+1}}$$
$$=\frac{1}{\left(\sqrt{n-1}+\sqrt{n+1}\right)}\times\frac{\sqrt{n-1}-\sqrt{n+1}}{\sqrt{n-1}-\sqrt{n+1}}$$
$$=\frac{\sqrt{n-1}-\sqrt{n+1}}{n-1-n-1}$$
$$=\frac{\sqrt{n+1}-\sqrt{n-1}}{2}$$

This implies that $\sqrt{n+1}-\sqrt{n-1}$ is also rational. Thus, $\sqrt{n+1}$ and $\sqrt{n-1}$ must both be rational (why?). Hence, $\sqrt{n+1}$ and $\sqrt{n-1}$ are integers, and the terms under the square-root signs must both be perfect squares. This is not possible as any two perfect squares differ by at least by 3. Therefore, there is no positive integer *n* such that $\sqrt{n-1}+\sqrt{n+1}$ is rational.

S3. (B). Let

$$x=\sqrt[3]{2}+3\sqrt[3]{4}$$
$$\Rightarrow\quad x^3=\left(\sqrt[3]{2}+3\sqrt[3]{4}\right)^2$$
$$=\begin{cases}\left(\sqrt[3]{2}\right)^3+\left(3\sqrt[3]{4}\right)^3\\ +3\left(\sqrt[3]{2}\right)\left(3\sqrt[3]{4}\right)\left(\sqrt[3]{2}+3\sqrt[3]{4}\right)\end{cases}$$
$$=2+108+9\sqrt[3]{8}x$$
$$=110+18x$$
$$\Rightarrow\quad x^3-18x-110=0$$

S4. (B). Refer to the figure below to understand how each length can be calculated using the Pythagoras theorem:



$$OQ = \sqrt{1^2 + 3^2} = \sqrt{10}$$
$$OR = \sqrt{OX^2 + XR^2}$$
$$= \sqrt{3^2 + 4^2} = 5$$
$$OS = \sqrt{OY^2 + YS^2}$$
$$= \sqrt{4^2 + 4^2} = \sqrt{32}$$
$$OT = \sqrt{OZ^2 + ZT^2}$$
$$= \sqrt{4^2 + 6^2} = \sqrt{52}$$

We see that OQ, OS and OT are all irrational, while OR is rational.

S5. (A), (C) and (E). By rationalizing the denominator of *y*, we note the following:

$$y = \frac{1}{a + b\sqrt{2}} \times \frac{a - b\sqrt{2}}{a - b\sqrt{2}}$$
$$= \frac{a - b\sqrt{2}}{a^2 - 2b^2} = a - b\sqrt{2}$$

It can now easily be deduced that the following terms will be rational: $x + y,\ x^2 + y^2,\ x^3 + y^3$

On the other hand, the following terms will be irrational, since they will contain factors of $\sqrt{2}$ in them: $x - y,\ x^2 - y^2$

Verifying these facts is left to the student as an exercise.

S6. (D). We have:

$$\frac{1}{x} = \frac{1}{3 + \sqrt{8}} = \frac{1}{3 + \sqrt{8}} \times \frac{3 - \sqrt{8}}{3 - \sqrt{8}}$$
$$= \frac{3 - \sqrt{8}}{9 - 8} = 3 - \sqrt{8}$$

Thus, we have:

$$x + \frac{1}{x} = 3 + \sqrt{8} + 3 - \sqrt{8}$$
$$\Rightarrow \quad x + \frac{1}{x} = 6$$

On squaring both sides, we get

$$\left(x + \frac{1}{x}\right)^2 = (6)^2$$
$$\Rightarrow \quad x^2 + \frac{1}{x^2} + 2(x)\left(\frac{1}{x}\right) = 36$$
$$\Rightarrow \quad x^2 + \frac{1}{x^2} = 36 - 2$$
$$\Rightarrow \quad x^2 + \frac{1}{x^2} = 34$$

Again squaring both sides, we get

$$\left(x^2+\frac{1}{x^2}\right)^2=(34)^2$$

$$\Rightarrow \quad x^4+\frac{1}{x^4}+2=1156$$

$$\Rightarrow \quad x^4+\frac{1}{x^4}=1154$$

$$\therefore \quad \frac{\left(x^4+\frac{1}{x^4}\right)}{\left(x^2+\frac{1}{x^2}\right)}=\frac{1154}{34}=\frac{577}{17}$$

Verify that this lies between 30 and 40.

One or More than Once Choices Correct

S7. (D) is the only correct option. The number of people in the world is obviously finite, and today, that number is less than 10 billion or $10,000,000,000$. You might think that the number of sand grains on a beach or the number of water molecules on Earth will be infinite. However, even these numbers are finite. If you had enough patience, you could count the number of sand grains on the beach one by one. You may not be able to count water molecules, but you can indirectly conclude that the number of water molecules on Earth must be finite, even though that number will be incredibly large, but it will be finite because the only other option is not possible. If the number of water molecules on Earth was infinite, then the mass of all that water would have been infinite, while we do know that this is not true, the Earth as a whole as finite (though very, very large) mass. However, the number of points on a line segment, no matter how small, *will always be infinite*. Can you visualize this fact? Take help of the Zooming Thought Experiment.

S8. (C) and (D). Explanations:

(A) False. The set of real numbers *contains* the set of irrational numbers, so once the Real set is plotted on a line, we get a continuous number line with no holes.

(B) False. Between any two distinct rational numbers, no matter how close, there will be infinitely many irrational numbers.

(C) True. Between any two distinct irrational numbers, no matter how close, there will be infinitely many rational numbers.

(D) True. Every real number can be thought of as a point on the horizontal number line in a two-dimensional plane, and hence is a complex number. Algebraically, any real number x can be written in complex form as $x=x+i0$.

S9. (B). Explanations:

(A) False. Irrational numbers which can be expressed as roots of rational numbers are only a particular class of irrational numbers. There are (infinitely many) irrational numbers which cannot be expressed this way. π is one example.

(B) True. For example: $\left(\sqrt{2}\right)+\left(1-\sqrt{2}\right)=1$

(C) False. The position of an irrational number on the number line has nothing to do with its decimal representation. For example, the square-root of 2 (which is an irrational number) can be exactly constructed on the number line.

(D) False. The decimal representation of an irrational number is non-terminating, but that of a rational number could be terminating, or non-terminating but repeating.

(E) False. We have only been given the decimal representation to ten decimal digits. The repetition of decimal digits could start after ten decimal digits, and thus the number could be rational.

S10. (E) is the only correct option. Point-wise explanations:

(A) INCORRECT. The product of two rational numbers will always be rational. If the two numbers are $\frac{p}{q}$ and $\frac{r}{s}$, then the product is $\frac{pr}{qs}$, which is obviously rational.

(B) INCORRECT. It is not necessary for the sum of two irrational numbers to be irrational. A simple example is: $\sqrt{2}+\left(-\sqrt{2}\right)=0$. Another example: we know that π is an irrational number, whose decimal representation is 3.14159...(non-terminating, non-repeating). Now, if we write π as the sum of 3 and the fractional part, we see that the fractional part must be irrational:

$$\pi = 3.14159... = \underset{\text{Rational}}{3} + \underset{\text{Irrational}}{0.14159...}$$

If we represent the fractional part by *f*, and consider the irrational number $x=-f$, we see that $\pi + x = 3$ which is a rational number.

(C) INCORRECT. Again, it is not necessary for the product of two irrational numbers to be irrational. For example:

$$\sqrt{2}\times\sqrt{2}=2,\ \pi\times\frac{1}{\pi}=1$$

(D) INCORRECT. This point is especially important to keep in mind. Using only the four basic arithmetic operations, it is not possible to construct irrational numbers from rational numbers. No matter what rational numbers you add, subtract, multiply or divide, you will always end up with rational numbers. In mathematical terminology, we say that the set of Rational Numbers is *closed with respect to these four operations*. This means that performing these operations in the Rational set will give you values within the Rational set only, and not outside it.

(E) CORRECT. We have already seen in the explanations of the first two options that it is possible to construct rational numbers from irrational numbers using the four basic arithmetic operations. Therefore, the set of Irrational Numbers is *not closed* with respect to these four operations.

S11. (D) and (E). No position of Theta can be represented using a rational number, because every position of Theta will numerically be equal to π divided by some power of 2, and so it will be irrational. Also, the sequence of steps in Theta's journey is infinitely long, since no matter how close he reaches to his final position, there are still infinitely many steps remaining, so mathematically speaking, Theta will never reach his final destination. This also means that option (D) is true, since the last 0.0000000001% of his journey will have infinitely many steps remaining, while the first 99.9999999999% will have finitely many steps.

S12. (A), (B) and (C). We have:

$$a+b\sqrt[3]{p}+c\sqrt[3]{p^2}=0 \quad ...(i)$$

Multiplying both sides by $\sqrt[3]{p}$, we get

$$a\sqrt[3]{p}+b\sqrt[3]{p^2}+cp=0 \quad ...(ii)$$

Now, multiplying equation (i) by *b* and equation (ii) by *c*, we get

$$ab+b^2\sqrt[3]{p}+bc\sqrt[3]{p^2}=0 \qquad ...(iii)$$

$$ac\sqrt[3]{p}+bc\sqrt[3]{p^2}+c^2p=0 \qquad ...(iv)$$

Subtracting equation (iv) from equation (iii), we get:

$$\left(b^2-ac\right)\sqrt[3]{p}+ab-c^2p=0 \quad ...(v)$$

Since $\sqrt[3]{p}$ is irrational, we must have $b^2-ac=0$ and $ab-c^2p=0$:

$$\therefore \quad ab-c^2p=0$$

$$\Rightarrow \quad c^2p=ab$$

$$\Rightarrow \quad c^4p^2=a^2b^2=a^3c$$

$$\Rightarrow \quad c\left(c^3p^2-a^3\right)=0$$

If $c\neq 0$, then we get $p^2=\frac{a^3}{c^3}$ which is not true as $\sqrt[3]{p}$ is irrational. Therefore, $c=0$ which in turn implies that $a=0$ and $b=0$. Hence, (A), (B) and (C) are the correct options.

S13. (D) is the only correct option. Point-wise explanations:

(A) False. Here's a counter-example:

$$\left(\sqrt[4]{2}\right)+\left(-\sqrt[4]{2}\right)=0$$

$$\text{but } \left(\sqrt[4]{2}\right)\left(-\sqrt[4]{2}\right)=-\sqrt{2}$$

(B) False. The set of Irrational Numbers is infinitely larger than the set of Rational Numbers.

(C) False. For $\frac{p}{q}$ to be a rational number, *p* and *q* must be integers, and *q* must be non-zero.

(D) True. We can find an irrational *x* just to the right of 0, and an irrational *y* just to the left of 0, such that $x-y$ is as small as we want it to be.

S14. (A) and (C). Point-wise explanations:

(A) CORRECT. $\sqrt{a+b}$ might be rational. For example, consider $\sqrt{2}$ and $\sqrt{7}$. Both of them are irrational. However, $\sqrt{2+7}=\sqrt{9}=3$ is rational. On the other hand, both $\sqrt{2}$ and $\sqrt{5}$ are irrational, and $\sqrt{2+5}=\sqrt{7}$ is also irrational.

(B) INCORRECT. $\sqrt{a-b}$ might be rational. For example, $\sqrt{11}$ and $\sqrt{2}$ are both irrational, but $\sqrt{11-2}=\sqrt{9}=3$ is rational.

(C) CORRECT. Consider $\sqrt{3}$ and $\sqrt{4}$. Both are irrational but $\sqrt{3^2+4^2}=\sqrt{25}=5$ is rational. On the other hand, think of $\sqrt{2}$ and $\sqrt{3}$. Both of them are irrational, and so is $\sqrt{2^2+3^2}=\sqrt{13}$.

(D) INCORRECT. $\left(\sqrt{a}+\sqrt{b}\right)^2$ might not always be rational. If you expand the square expression, you have

$$\begin{aligned}\left(\sqrt{a}+\sqrt{b}\right)^2 &= \left(\sqrt{a}\right)^2+\left(\sqrt{b}\right)^2+2\left(\sqrt{a}\right)\left(\sqrt{b}\right)\\ &= a+b+2\sqrt{ab}\end{aligned}$$

If $\sqrt{ab}$ is irrational, then so is $\left(\sqrt{a}+\sqrt{b}\right)^2$. For example,

$$\left(\sqrt{2}+\sqrt{3}\right)^2=2+3+2\sqrt{6}=5+2\sqrt{6} \quad \text{(irrational)}$$

On the other hand,

$$\left(\sqrt{2}+\sqrt{8}\right)^2=2+8+2\sqrt{16}=18 \quad \text{(rational)}$$

S15. (A), (B), (C) and (D). We know that $\sqrt{2}$ and $\sqrt{5}$ are irrational numbers and so is $\sqrt{2}+\sqrt{5}$ (showing this is left to you as an exercise). Now, we have:

$$\begin{aligned}\left(\sqrt{2}+\sqrt{5}\right)^2 &= \left(\sqrt{2}\right)^2+\left(\sqrt{5}\right)^2+2\left(\sqrt{2}\right)\left(\sqrt{5}\right)\\ &= 2+5+2\sqrt{10}\\ &= 7+2\sqrt{10}\end{aligned}$$

$$\begin{aligned}\Rightarrow \left(\sqrt{2}+\sqrt{5}\right)^4 &= \left(7+2\sqrt{10}\right)^2\\ &= 49+40+28\sqrt{10}\\ &= 89+28\sqrt{10}\end{aligned}$$

It is easy to see that if we keep on squaring this, we will always be left with an irrational term in the resulting expression, which means that the expressions in (B), (C) and (D) are all irrational.

S16. (A) and (B). These are irrational because if these were rational, then $\left(\sqrt{x}\right)^2$ and $\left(\sqrt[3]{x}\right)^3$, that is, *x*, would have been rational, which is not the case.

Option (C).If we take $x=\sqrt{2}$ then $x^2=2$ which is rational. Hence, x^2 may be rational or irrational.

Option (D). If we take $x=2^{\frac{1}{\pi}}$ then $x^{\pi}=2$, which is rational. Hence, x^{π} may be rational or irrational.

Option (E). If we take $x=5^{\sqrt{2}}$ then $x^{\frac{1}{\sqrt{2}}}=5$, which is rational. Hence, $x^{\left(\frac{1}{\sqrt{2}}\right)}$ may be rational or irrational.

We see that (A) and (B) are the only correct options.

S17. (B) and (C). Self-exercise. (A) and (D) will always be irrational. For (B) and (C), try to construct examples to show that these can be rational.

S18. (A) and (B). We have:

$$A_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Putting n = 0, we get

$$A_0 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(\frac{1-\sqrt{5}}{2}\right)^0\right]$$

$$= \frac{1}{\sqrt{5}}[1-1] \quad \left[x^0 = 1 \ \forall \ x\right]$$

$$= 0$$

Putting n = 1, we get

$$A_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{2\sqrt{5}}{2}\right] = 1$$

Putting n = 2, we get

$$A_2 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{1+5+2\sqrt{5}-1-5+2\sqrt{5}}{4}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{4\sqrt{5}}{4}\right] = 1$$

Putting n = 3, we get

$$A_3 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^3 - \left(\frac{1-\sqrt{5}}{2}\right)^3\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{\begin{matrix}(1)^3 + \left(\sqrt{5}\right)^3 + 3\sqrt{5} + 3\left(\sqrt{5}\right)^2 \\ -(1)^3 + \left(\sqrt{5}\right)^3 + 3\left(\sqrt{5}\right) - 3(5)^2\end{matrix}}{8}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{2\left(\sqrt{5}\right)^3 + 6\sqrt{5}}{8}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{10\sqrt{5}+6\sqrt{5}}{8}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{16\sqrt{5}}{8}\right] = 2$$

Clearly, options (A) and (B) are the correct options.

Integer Answers

S19. By rationalizing the denominator of the second term, we note the following:

$$\frac{1}{\sqrt{2}+1}=\frac{1}{\sqrt{2}+1}\times\frac{\sqrt{2}-1}{\sqrt{2}-1}=\sqrt{2}-1$$

Now, we square the two terms:

$$\left(\sqrt{2}+1\right)^2=2+1+2\sqrt{2}=3+2\sqrt{2}$$

$$\left(\sqrt{2}-1\right)^2=2+1-2\sqrt{2}=3-2\sqrt{2}$$

And then we square them again:

$$\left(\sqrt{2}+1\right)^4=\left(3+2\sqrt{2}\right)^2$$

$$=9+8+12\sqrt{2}=17+12\sqrt{2}$$

And:

$$\left(\sqrt{2}-1\right)^4=\left(3-2\sqrt{2}\right)^2$$

$$=9+8-12\sqrt{2}=17-12\sqrt{2}$$

Finally, adding the two, we see that the required value is

$$\left(17+12\sqrt{2}\right)+\left(17-12\sqrt{2}\right)=34$$

S20. We rationalize the denominator in the second term:

$$\frac{1}{\sqrt{3}+1}=\frac{1}{\sqrt{3}+1}\times\frac{\sqrt{3}-1}{\sqrt{3}-1}=\frac{\sqrt{3}-1}{2}$$

Now, we evaluate the two terms separately:

$$2\left(\sqrt{3}+1\right)^2=2\left(3+1+2\sqrt{3}\right)$$

$$=2\left(4+2\sqrt{3}\right)=8+4\sqrt{3}$$

And:

$$\frac{1}{\left(\sqrt{3}+1\right)^4}=\left(\frac{\sqrt{3}-1}{2}\right)^4$$

$$=\left(\left(\frac{\sqrt{3}-1}{2}\right)^2\right)^2=\left(2-\sqrt{3}\right)^2 \quad \text{(how?)}$$

$$=7-4\sqrt{3}$$

Finally, adding the two results, we see that the required value is 15.

S21. To rationalize the denominator of the given expression, we proceed as follows:

$$\frac{1}{\sqrt{5}+\sqrt{3}-\sqrt{8}}\times\frac{\sqrt{5}+\sqrt{3}+\sqrt{8}}{\sqrt{5}+\sqrt{3}+\sqrt{8}}$$

$$=\frac{\sqrt{5}+\sqrt{3}+\sqrt{8}}{\left(\sqrt{5}+\sqrt{3}\right)^2-\left(\sqrt{8}\right)^2}$$

$$=\frac{\sqrt{5}+\sqrt{3}+\sqrt{8}}{8+2\sqrt{15}-8}$$

$$=\frac{\sqrt{5}+\sqrt{3}+\sqrt{8}}{2\sqrt{15}}$$

Now, we do a second rationalization step:

$$\frac{\sqrt{5}+\sqrt{3}+\sqrt{8}}{2\sqrt{15}}\times\frac{\sqrt{15}}{\sqrt{15}}$$

$$=\frac{\sqrt{75}+\sqrt{45}+\sqrt{120}}{30}$$

$$=\frac{5\sqrt{3}+3\sqrt{5}+2\sqrt{30}}{30}$$

$$=\frac{1}{6}\sqrt{3}+\frac{1}{10}\sqrt{5}+\frac{1}{15}\sqrt{30}$$

Comparing with the expression given in the question, we have:

$$a=\frac{1}{6},\ b=\frac{1}{10},\ c=\frac{1}{15}$$

$$\Rightarrow\ 12(a+b+c)=12\left(\frac{1}{6}+\frac{1}{10}+\frac{1}{15}\right)$$

$$=12\left(\frac{10}{30}\right)=4$$

S22. We have:

$$\frac{\sqrt{2}}{\sqrt{2+\sqrt{3}}-\sqrt{2-\sqrt{3}}}$$

$$=\frac{\sqrt{2}}{\sqrt{2+\sqrt{3}}-\sqrt{2-\sqrt{3}}}\times\frac{\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}}{\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}}$$

$$=\frac{\sqrt{2}\left(\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}\right)}{\left(\sqrt{2+\sqrt{3}}\right)^2-\left(\sqrt{2-\sqrt{3}}\right)^2}$$

$$=\frac{\sqrt{2}\left(\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}\right)}{2\sqrt{3}}$$

Now, we will calculate the square root of $2+\sqrt{3}$. Let

$$\sqrt{2+\sqrt{3}}=\sqrt{a}+\sqrt{b}$$

$$\Rightarrow \left(\sqrt{2+\sqrt{3}}\right)^2=\left(\sqrt{a}+\sqrt{b}\right)^2$$

$$\Rightarrow 2+\sqrt{3}=a+b+2\sqrt{ab}$$

$$\Rightarrow a+b=2,\ 2\sqrt{ab}=\sqrt{3}$$

$$\Rightarrow a+b=2, ab=\frac{3}{4}$$

$$\Rightarrow a=\frac{1}{2}, b=\frac{3}{2}$$

$$\therefore \sqrt{2+\sqrt{3}}=\sqrt{\frac{1}{2}}+\sqrt{\frac{3}{2}}=\frac{1+\sqrt{3}}{\sqrt{2}}$$

Similarly,

$$\sqrt{2-\sqrt{3}}=\frac{-1+\sqrt{3}}{\sqrt{2}}$$

Thus, we have:

$$\frac{\sqrt{2}\left(\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}\right)}{2\sqrt{3}}$$

$$=\frac{\sqrt{2}\left(\frac{1+\sqrt{3}}{\sqrt{2}}+\frac{-1+\sqrt{3}}{\sqrt{2}}\right)}{2\sqrt{3}}$$

$$=\frac{2\sqrt{3}}{2\sqrt{3}}=1$$

$$\Rightarrow \frac{\sqrt{2}}{\sqrt{2+\sqrt{3}}-\sqrt{2-\sqrt{3}}}=1$$

S23. The correct answer is 3. We have:

$$\frac{4\sqrt{7}}{\sqrt{16+6\sqrt{7}}-\sqrt{16-6\sqrt{7}}}$$

$$=\frac{4\sqrt{7}}{\sqrt{16+6\sqrt{7}}-\sqrt{16-6\sqrt{7}}}\times\frac{\left(\sqrt{16+6\sqrt{7}}+\sqrt{16-6\sqrt{7}}\right)}{\left(\sqrt{16+6\sqrt{7}}+\sqrt{16-6\sqrt{7}}\right)}$$

$$=\frac{4\sqrt{7}\left(\sqrt{16+6\sqrt{7}}-\sqrt{16-6\sqrt{7}}\right)}{\left(\sqrt{16+6\sqrt{7}}\right)^2-\left(\sqrt{16-6\sqrt{7}}\right)^2}$$

$$=\frac{4\sqrt{7}\left(\sqrt{16+6\sqrt{7}}-\sqrt{16-6\sqrt{7}}\right)}{16+6\sqrt{7}-16+6\sqrt{7}}$$

$$= \frac{4\sqrt{7}\left(\sqrt{16+6\sqrt{7}}-\sqrt{16-6\sqrt{7}}\right)}{12\sqrt{7}}$$

$$= \frac{\sqrt{16+6\sqrt{7}}+\sqrt{16-6\sqrt{7}}}{3}$$

Now, we will calculate the square root of $16+6\sqrt{7}$. Let

$$\sqrt{16+6\sqrt{7}} = \sqrt{a}+\sqrt{b}$$

$$\Rightarrow \left(\sqrt{16+6\sqrt{7}}\right)^2 = \left(\sqrt{a}+\sqrt{b}\right)^2$$

$$\Rightarrow 16+6\sqrt{7} = a+b+2\sqrt{ab}$$

$$\Rightarrow a+b=16, \sqrt{ab}=3\sqrt{7}$$

$$\Rightarrow a+b=16, ab=63$$

$$\Rightarrow a=9, b=7$$

$$\therefore \sqrt{16+6\sqrt{7}} = 3+\sqrt{7}$$

Similarly, $\sqrt{16-6\sqrt{7}} = 3-\sqrt{7}$. Thus, we have:

$$\frac{\sqrt{16+6\sqrt{7}}+\sqrt{16-6\sqrt{7}}}{3}$$

$$= \frac{3+\sqrt{7}+3-\sqrt{7}}{3} = 3$$

S24. First, we will simplify each term one by one. We have:

$$\frac{1}{\sqrt{2}+1} = \frac{1}{\sqrt{2}+1}\times\frac{\sqrt{2}-1}{\sqrt{2}-1}$$

$$= \frac{\sqrt{2}-1}{2-1} = \sqrt{2}-1$$

$$\frac{1}{\sqrt{3}+\sqrt{2}} = \frac{1}{\sqrt{3}+\sqrt{2}}\times\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}$$

$$= \frac{\sqrt{3}-\sqrt{2}}{3-2} = \sqrt{3}-\sqrt{2}$$

$$\frac{1}{\sqrt{4}+\sqrt{3}} = \frac{1}{\sqrt{4}+\sqrt{3}}\times\frac{\sqrt{4}-\sqrt{2}}{\sqrt{4}-\sqrt{3}}$$

$$= \frac{\sqrt{4}-\sqrt{3}}{4-3} = \sqrt{4}-\sqrt{3}$$

$$\vdots\ \vdots\ \vdots$$

$$\frac{1}{\sqrt{9}+\sqrt{8}} = \frac{1}{\sqrt{9}+\sqrt{8}}\times\frac{\sqrt{9}-\sqrt{8}}{\sqrt{9}-\sqrt{8}}$$

$$= \frac{\sqrt{9}-\sqrt{8}}{9-8} = \sqrt{9}-\sqrt{8}$$

Thus, the required sum is equal to

$$\begin{aligned}&\left(\sqrt{2}-1\right)+\left(\sqrt{3}-\sqrt{2}\right)\\&+\left(\sqrt{4}-\sqrt{3}\right)+\ldots+\left(\sqrt{9}-\sqrt{8}\right)\\&=-1+\sqrt{9}\\&=-1+3\\&=2\end{aligned}$$

S25. First, we will simplify each term one by one. We have:

$$\begin{aligned}\frac{2\sqrt{6}}{\sqrt{2}+\sqrt{3}}&=\frac{2\sqrt{6}\left(\sqrt{2}-\sqrt{3}\right)}{\left(\sqrt{2}+\sqrt{3}\right)\left(\sqrt{2}-\sqrt{3}\right)}\\&=\frac{2\sqrt{6}\left(\sqrt{2}-\sqrt{3}\right)}{2-3}\\&=6\sqrt{2}-4\sqrt{3}\end{aligned}$$

$$\begin{aligned}\frac{6\sqrt{2}}{\sqrt{6}+\sqrt{3}}&=\frac{6\sqrt{2}\left(\sqrt{6}-\sqrt{3}\right)}{\left(\sqrt{6}+\sqrt{3}\right)\left(\sqrt{6}-\sqrt{3}\right)}\\&=\frac{6\sqrt{2}\left(\sqrt{6}-\sqrt{3}\right)}{6-3}\\&=4\sqrt{3}-2\sqrt{6}\end{aligned}$$

$$\begin{aligned}\frac{8\sqrt{3}}{\sqrt{6}+\sqrt{2}}&=\frac{8\sqrt{3}\left(\sqrt{6}-\sqrt{2}\right)}{\left(\sqrt{6}+\sqrt{2}\right)\left(\sqrt{6}-\sqrt{2}\right)}\\&=\frac{8\sqrt{3}\left(\sqrt{6}-\sqrt{2}\right)}{6-2}\\&=6\sqrt{2}-4\sqrt{6}\end{aligned}$$

Thus, the given expression is equivalent to:

$$\begin{aligned}&6\sqrt{2}-4\sqrt{3}+4\sqrt{3}-2\sqrt{6}-6\sqrt{2}+4\sqrt{6}\\&=2\sqrt{6}\approx 4.90\end{aligned}$$

The greatest integer less than this value is 4.

S26. If we denote the value of this expression by *x*, we have

$$x=\sqrt{6+x}\qquad(\text{how ?})$$

By observation, we can say that $x=3$ satisfies this relation, which means that the value of the given expression is 3.

S27. We have

$$\begin{cases}10+\sqrt{24}+\sqrt{60}+\sqrt{40}\\ \qquad = \\ \left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2\end{cases}$$

Squaring the second expression and comparing both sides, we have (note this carefully):

$$\begin{cases} x+y+z=10 & ...(1) \\ 2\sqrt{xy}=\sqrt{24} & ...(2) \\ 2\sqrt{yz}=\sqrt{60} & ...(3) \\ 2\sqrt{xz}=\sqrt{40} & ...(4) \end{cases}$$

Multiplying the last three expressions, we have

$$\begin{aligned} 8xyz &= \sqrt{24}\times\sqrt{60}\times\sqrt{40} \\ &= 2\sqrt{6}\times 2\sqrt{15}\times 2\sqrt{10} \\ &= 8\sqrt{900} \\ &= 8\times 30 \\ \Rightarrow\ xyz &= 30 \\ \Rightarrow\ \sqrt{xyz} &= \sqrt{30} \qquad ...(5) \end{aligned}$$

Dividing (5) by (3), we have

$$\frac{\sqrt{x}}{2}=\frac{1}{\sqrt{2}}$$

$$\Rightarrow\ x=2$$

Similarly, *y* and *z* can be found out as 3 and 5. Thus, $y+z-x=6$.

S28. We note that

$$\sqrt{63}=\sqrt{9\times 7}=3\sqrt{7}$$

$$\sqrt{56}=\sqrt{8\times 7}=2\left(\sqrt{2}\right)\left(\sqrt{7}\right)$$

Now, we have

$$\begin{aligned} \sqrt{\sqrt{63}+\sqrt{57}} &= \sqrt{3\sqrt{7}+2\left(\sqrt{2}\right)\left(\sqrt{7}\right)} \\ &= \sqrt{\sqrt{7}\left(3+2\sqrt{2}\right)} \\ &= \sqrt[4]{7}\sqrt{3+2\sqrt{2}} \end{aligned}$$

You can easily show that the square root of $3+2\sqrt{2}$ is $\sqrt{2}+1$, and thus,

$$\sqrt{\sqrt{63}+\sqrt{57}}=\sqrt[4]{7}\left(\sqrt{2}+1\right)$$

$$\Rightarrow\quad a=7,\ b=2,\ c=1$$

$$\Rightarrow\quad a+2\left(b+c\right)=13$$

S29. We will simplify each term one by one. We have:

$$\frac{7\sqrt{3}}{\sqrt{10}+\sqrt{3}}=\frac{7\sqrt{3}\left(\sqrt{10}-\sqrt{3}\right)}{\left(\sqrt{10}+\sqrt{3}\right)\left(\sqrt{10}-\sqrt{3}\right)}$$
$$=\frac{7\sqrt{3}\left(\sqrt{10}-\sqrt{3}\right)}{10-3}=\sqrt{30}-3$$
$$\frac{2\sqrt{5}}{\sqrt{6}+\sqrt{5}}=\frac{2\sqrt{5}\left(\sqrt{6}-\sqrt{5}\right)}{\left(\sqrt{6}+\sqrt{5}\right)\left(\sqrt{6}-\sqrt{5}\right)}$$
$$=\frac{2\sqrt{5}\left(\sqrt{6}-\sqrt{5}\right)}{6-5}=2\sqrt{30}-10$$
$$\frac{3\sqrt{2}}{\sqrt{15}+3\sqrt{2}}=\frac{3\sqrt{2}\left(\sqrt{15}-3\sqrt{2}\right)}{\left(\sqrt{15}+3\sqrt{2}\right)\left(\sqrt{15}-3\sqrt{2}\right)}$$
$$=\frac{3\sqrt{2}\left(\sqrt{15}-3\sqrt{2}\right)}{15-18}=-\sqrt{30}+6$$

Thus, the required sum is equal to

$$\left(\sqrt{30}-3\right)-\left(2\sqrt{30}-10\right)-\left(-\sqrt{30}+6\right)$$
$$=2\sqrt{30}-9-2\sqrt{30}+10$$
$$=1$$

Miscellaneous

S30. Take a prime number k, and suppose that its square-root can be expressed as a rational number: $\sqrt{k}=\frac{p}{q}$. We suppose that there is no common factor between p and q. Now, we will arrive at a contradiction, as we did in the case of the square-roots of $\sqrt{2}$ and $\sqrt{3}$:

$$k=\frac{p^2}{q^2}\Rightarrow p^2=kq^2$$

The right side is a multiple of k, which means that the square of p, and hence p itself, must be a multiple of k. Thus,

$$p=mk\Rightarrow (mk)^2=kq^2$$
$$\Rightarrow q^2=km^2$$

The right side is a multiple of k again, which means that the square of q, and hence q itself, must be a multiple of k. Thus, we have shown that both p and q are multiples of k, which is a contradiction (we assumed no common factors between them). This means that the square-root of k cannot be written as a rational number.

As an exercise, show why we cannot apply the same approach to non-prime natural numbers (that is, the square-root of every non-prime natural number will not necessarily be irrational).

S31. We will construct particular examples to demonstrate this fact. If x is irrational, then

$$y=\frac{x}{2}$$

is also an irrational number such that $0<y<x$. If x is rational, then

$$y=\frac{x}{\sqrt{2}}$$

is an irrational number such that

$$0<y<x \text{ as } \sqrt{2}>1$$

S32. Clearly, $\frac{p+r}{q+s}$ will be a well-defined rational number as $q+s$ is a positive integer (that is, it cannot be zero). We have:

$$\frac{p}{q} < \frac{r}{s}$$

$$\Rightarrow \quad ps < qr \qquad \text{...(i)}$$

Now,

$$\frac{p}{q} < \frac{p+r}{q+s}$$

$$\Rightarrow \quad p(q+s) < q(p+r)$$

$$\Rightarrow \quad pq + ps < pq + qr$$

$$\Rightarrow \quad ps < qr$$

which is true because of (1). Also,

$$\frac{p+r}{q+s} < \frac{r}{s}$$

$$\Rightarrow \quad (p+r)s < (q+s)r$$

$$\Rightarrow \quad ps + rs < qr + sr$$

$$\Rightarrow \quad ps < qr$$

which is true because of (1). Thus,

$$\frac{p}{q} < \frac{p+r}{q+s} < \frac{r}{s}$$

S33. Self-exercise.

S34. *x* is irrational. Self-exercise.

S35. We have:

$$\sqrt{\frac{2+\sqrt{3}}{2-\sqrt{3}}} = \sqrt{\frac{2+\sqrt{3}}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}}}$$

$$= \sqrt{\frac{\left(2+\sqrt{3}\right)^2}{(2)^2 - \left(\sqrt{3}\right)^2}} = 2+\sqrt{3}$$

$$\approx 2 + 1.732 = 3.732$$

S36. We have:

$$x = \frac{1}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}}$$

$$= \frac{2+\sqrt{3}}{(2)^2 - \left(\sqrt{3}\right)^2}$$

$$= \frac{2+\sqrt{3}}{4-3} = 2+\sqrt{3}$$

Now,

$$x^2 = \left(2+\sqrt{3}\right)^2$$
$$= (2)^2 + \left(\sqrt{3}\right)^2 + 2(2)\sqrt{3}$$
$$= 5 + 4\sqrt{3}$$
$$x^3 = \left(2+\sqrt{3}\right)^3$$
$$= (2)^3 + \left(\sqrt{3}\right)^3 + 3(2)^2\left(\sqrt{3}\right) + 3(2)\left(\sqrt{3}\right)^2$$
$$= 8 + 3\sqrt{3} + 12\sqrt{3} + 18$$
$$= 26 + 15\sqrt{3}$$

Thus, we have:

$$x^3 - 2x^2 - 7x + 5$$
$$= \left(26+15\sqrt{3}\right) - 2\left(5+4\sqrt{3}\right) - 7\left(2+\sqrt{3}\right) + 5$$
$$= 26 + 15\sqrt{3} - 10 - 8\sqrt{3} - 14 - 7\sqrt{3} + 5$$
$$= 7$$

S37. We have:

$$x = \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$$
$$= \frac{\left(\sqrt{5}-\sqrt{3}\right)\times\left(\sqrt{5}-\sqrt{3}\right)}{\left(\sqrt{5}+\sqrt{3}\right)\times\left(\sqrt{5}-\sqrt{3}\right)}$$
$$= \frac{\left(\sqrt{5}-\sqrt{3}\right)^2}{\left(\sqrt{5}\right)^2 - \left(\sqrt{3}\right)^2} = \frac{5+3-2\sqrt{15}}{5-3}$$
$$= \frac{8-2\sqrt{15}}{2} = 4 - \sqrt{15}$$

Thus,

$$x - 4 = -\sqrt{15}$$
$$\Rightarrow \ (x-4)^2 = \left(-\sqrt{15}\right)^2$$
$$\Rightarrow \ x^2 - 8x + 16 = 15$$
$$\Rightarrow \ x^2 - 8x + 16 - 15 = 0$$
$$\Rightarrow \ x^2 - 8x + 1 = 0$$

S38. First, we will simplify each term one by one. We have:

$$\frac{1}{3-\sqrt{8}}=\frac{1}{3-\sqrt{8}}\times\frac{3+\sqrt{8}}{3+\sqrt{8}}$$
$$=\frac{3+\sqrt{8}}{9-8}=3+\sqrt{8}$$

$$\frac{1}{\sqrt{8}-\sqrt{7}}=\frac{1}{\sqrt{8}-\sqrt{7}}\times\frac{\sqrt{8}+\sqrt{7}}{\sqrt{8}+\sqrt{7}}$$
$$=\frac{\sqrt{8}+\sqrt{7}}{8-7}=\sqrt{8}+\sqrt{7}$$

$$\frac{1}{\sqrt{7}-\sqrt{6}}=\frac{1}{\sqrt{7}-\sqrt{6}}\times\frac{\sqrt{7}+\sqrt{6}}{\sqrt{7}+\sqrt{6}}$$
$$=\frac{\sqrt{7}+\sqrt{6}}{7-6}=\sqrt{7}+\sqrt{6}$$

$$\frac{1}{\sqrt{6}-\sqrt{5}}=\frac{1}{\sqrt{6}-\sqrt{5}}\times\frac{\sqrt{6}+\sqrt{5}}{\sqrt{6}+\sqrt{5}}$$
$$=\frac{\sqrt{6}+\sqrt{5}}{6-5}=\sqrt{6}+\sqrt{5}$$

$$\frac{1}{\sqrt{5}-2}=\frac{1}{\sqrt{5}-2}\times\frac{\sqrt{5}+2}{\sqrt{5}+2}$$
$$=\frac{\sqrt{5}+2}{5-4}=\sqrt{5}+2$$

Thus, the given expression is equivalent to:

$$\left(3+\sqrt{8}\right)-\left(\sqrt{8}+\sqrt{7}\right)+\left(\sqrt{7}+\sqrt{6}\right)$$
$$-\left(\sqrt{6}+\sqrt{5}\right)+\left(\sqrt{5}+2\right)$$
$$=3+\sqrt{8}-\sqrt{8}-\sqrt{7}+\sqrt{7}$$
$$+\sqrt{6}-\sqrt{6}-\sqrt{5}+\sqrt{5}+2$$
$$=3+2=5$$

S39. We have:

$$\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2=\left(\sqrt{x}\right)^2+\left(\frac{1}{\sqrt{x}}\right)^2-2\left(\sqrt{x}\right)\left(\frac{1}{\sqrt{x}}\right)$$
$$=x+\frac{1}{x}-2=9+4\sqrt{5}+\frac{1}{9+4\sqrt{5}}-2$$
$$=9+4\sqrt{5}+\frac{9-4\sqrt{5}}{(9)^2-\left(4\sqrt{5}\right)^2}-2$$
$$=9+4\sqrt{5}+\frac{9-4\sqrt{5}}{81-80}-2$$
$$=9+4\sqrt{5}+9-4\sqrt{5}-2=16$$
$$\Rightarrow\quad \sqrt{x}-\frac{1}{\sqrt{x}}=4$$

S40. We have:

$$x = \frac{1}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{2-\sqrt{3}}{(2)^2 - \left(\sqrt{3}\right)^2}$$

$$= \frac{2-\sqrt{3}}{4-3} = 2-\sqrt{3}$$

Now,

$$x^2 = \left(2-\sqrt{3}\right)^2 = (2)^2 + \left(\sqrt{3}\right)^2 - 2(2)\left(\sqrt{3}\right)$$

$$= 7 - 4\sqrt{3}$$

$$x^3 = \left(2-\sqrt{3}\right)^3$$

$$= (2)^3 - \left(\sqrt{3}\right)^3 - 3(2)^2\left(\sqrt{3}\right) + 3(2)\left(\sqrt{3}\right)^2$$

$$= 8 - 3\sqrt{3} - 12\sqrt{3} + 18$$

$$= 26 - 15\sqrt{3}$$

Thus, we have:

$$2x^3 - 7x^2 - 2x + 1$$

$$= 2\left(26-15\sqrt{3}\right) - 7\left(7-4\sqrt{3}\right) - 2\left(2-\sqrt{3}\right) + 1$$

$$= 52 - 30\sqrt{3} - 49 + 28\sqrt{3} - 4 + 2\sqrt{3} + 1$$

$$= 0$$

S41. We have:

$$\frac{4}{3\sqrt{3}-2\sqrt{2}} + \frac{3}{3\sqrt{3}+2\sqrt{2}}$$

$$= \frac{4\left(3\sqrt{3}+2\sqrt{2}\right) + 3\left(3\sqrt{3}-2\sqrt{2}\right)}{\left(3\sqrt{3}-2\sqrt{2}\right)\left(3\sqrt{3}+2\sqrt{2}\right)}$$

$$= \frac{12\sqrt{3} + 8\sqrt{2} + 9\sqrt{3} - 6\sqrt{2}}{\left(3\sqrt{3}\right)^2 - \left(2\sqrt{2}\right)^2}$$

$$= \frac{21\sqrt{3} + 2\sqrt{2}}{27-8}$$

Now, we approximate the value of this expression:

$$\frac{21\sqrt{3} + 2\sqrt{2}}{19}$$

$$\approx \frac{21(1.732) + 2(1.414)}{19}$$

$$= \frac{36.372 + 2.828}{19}$$

$$= \frac{39.2}{19} \approx 2.06$$

The least integer greater than this value is 3.

S42. We have:

$$\frac{1}{x}=\frac{1}{3\sqrt{3}+\sqrt{26}}\times\frac{3\sqrt{3}-\sqrt{26}}{3\sqrt{3}-\sqrt{26}}$$

$$=\frac{3\sqrt{3}-\sqrt{26}}{\left(3\sqrt{3}\right)^2-\left(\sqrt{26}\right)^2}$$

$$=3\sqrt{3}-\sqrt{26}$$

Therefore:

$$x+\frac{1}{x}=3\sqrt{3}+\sqrt{26}+3\sqrt{3}-\sqrt{26}$$

$$\Rightarrow\ \frac{1}{2}\left(x+\frac{1}{x}\right)=\frac{1}{2}\left(6\sqrt{3}\right)=3\sqrt{3}$$