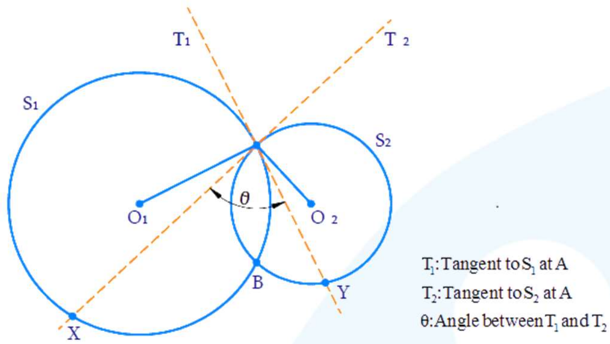
Circles and Tangents

Review Exercise Answers

Level-2

Miscellaneous

S1. Let us consider only the angle between the tangents at A. Consider the following figure, keeping in mind that the tangent to any circle is perpendicular to the radius through the point of contact.



Note that

$\angle O_1AO_2 = \angle O_1AX + \angle XAY + \angle YAC_2$

$= (90^\circ - \theta) + (\theta) + (90^\circ - \theta)$ (how ?)

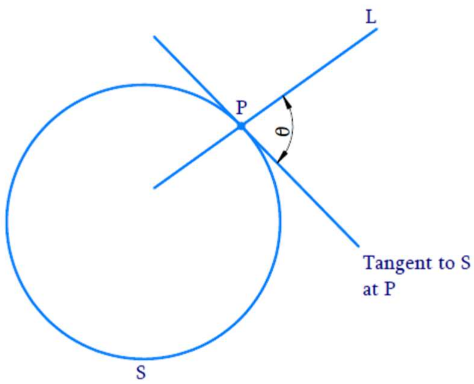
$= 180^\circ - \theta$

$\Rightarrow \quad \theta = 180^\circ - \angle O_1AO_2$

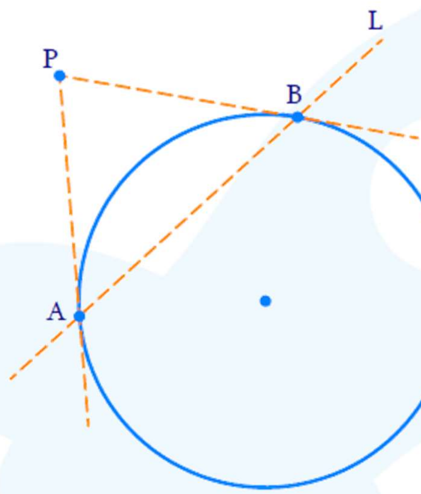
Thus, the value of the angle θ between the tangents at A is $180^\circ - \angle O_1AO_2$. Similarly, we can show that the value of the angle between the tangents at B must be $180^\circ - \angle O_1BO_2$. But $\angle O_1AO_2$ and $\angle O_1BO_2$ must be equal (why? Compare ΔO_1AO_2 and ΔO_1BO_2 - they are congruent by the SSS criterion).

This means that the angles between the tangents at A and B are equal.

S2. The statement of this problem might seem confusing to you. What do we mean by the angle at which a line cuts a circle? If a line L cuts a circle S at a point P, the angle between L and the tangent to S at P is the angle at which L cuts S:

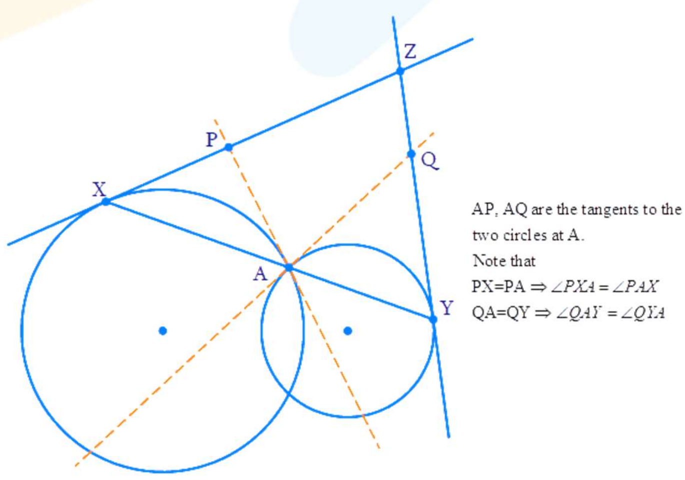


In the problem, we are given that a line cuts a circle at two points A and B, and we have to show that it cuts the circle at the same angle at each of these points. Consider the following figure:



Since $PA = PB, \Delta PAB$ is isosceles. Thus, $\angle PAB = \angle PBA$, which means that L is inclined to the two tangents at the same angle, i.e., L cuts the circle at the same angle at both A and B.

S3. We will make use of the fact that the two tangents drawn from ax external point to a circle are of equal lengths. Consider the following figure:

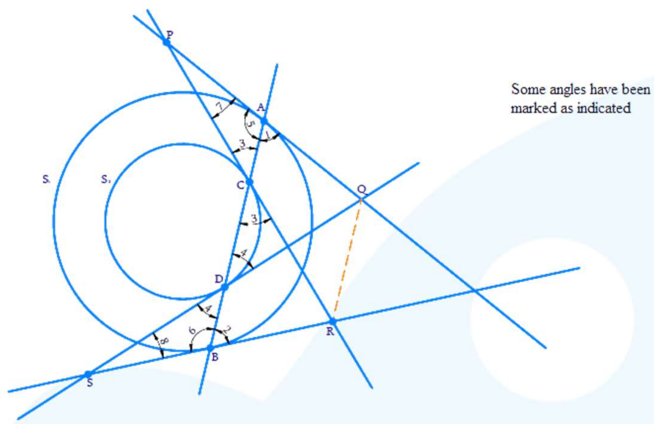


Now, we have:

$$\angle XZY = 180^\circ - (\angle PXA + \angle QYA)$$
$$= 180^\circ - (\angle PAX + \angle QAY)$$
$$= \angle PAQ$$

Thus, $\angle XZY$ is the same as the angle between the tangents at A.

S4. Observe the following figure carefully, and verify that each piece of information provided in the problem has been used correctly:



Since tangents drawn from an external point to a circle are of equal lengths, they are equally inclined to the chord of contact. Thus,

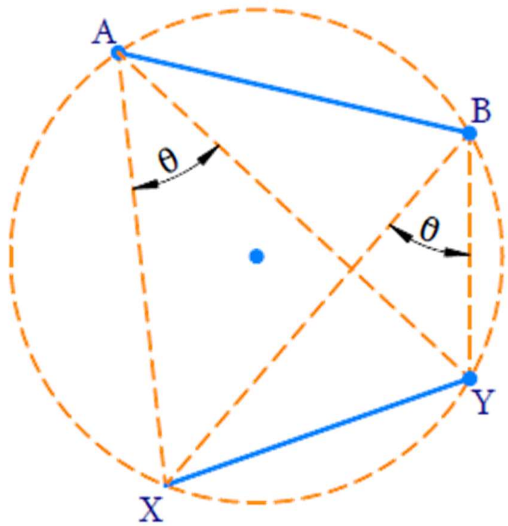
(i) $\angle 1 = \angle 2 \Rightarrow \angle 5 = \angle 6$

(ii) $\angle 3 = \angle 4$

This means that $\angle 7 = 180^\circ - (\angle 3 + \angle 5)$ will be equal to $\angle 8 = 180^\circ - (\angle 4 + \angle 6)$. That is, the angle that QR subtends at $P(\angle 7)$ is the same as the angle that QR subtends at $S(\angle 8)$.

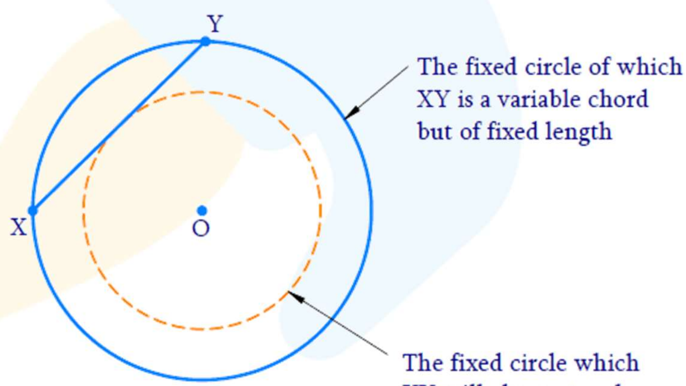
Clearly, P, Q, R and S must be concyclic points.

S5. Let A and B be the two fixed points, and let X and Y represent the end-points of the variable line segment of fixed length. Let θ be the fixed value of the angle that XY subtends at both A and B. Clearly, X and Y are concyclic with A and B, for any position and orientation of XY:

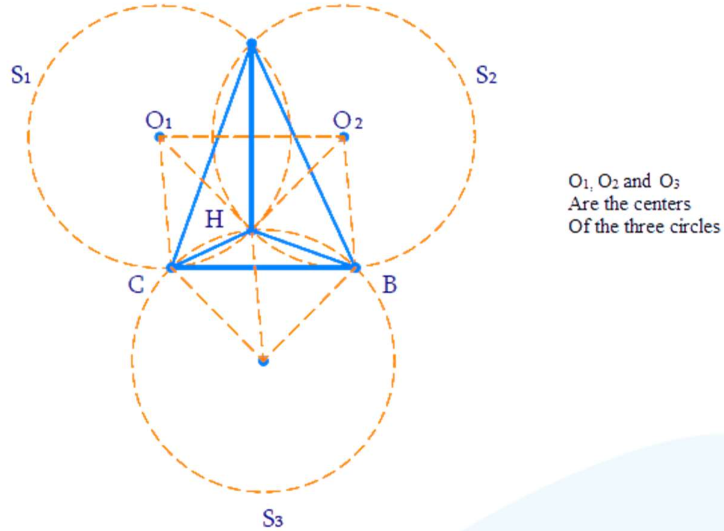


In fact, it is easy to see that this circle is fixed (since the length XY is fixed), and XY is a variable chord of this fixed circle. The center of this fixed circle can be determined by getting the perpendicular bisector of the fixed segment AB and the perpendicular bisector of any XY to intersect.

Now, since XY is a fixed-length variable chord of a fixed circle, (whose center we represent by O), we can think of it as a rod of fixed length sliding around inside a circle. Thus, its distance from the center O is fixed, which means that it will always touch a concentric circle whose radius is equal to its distance from O.

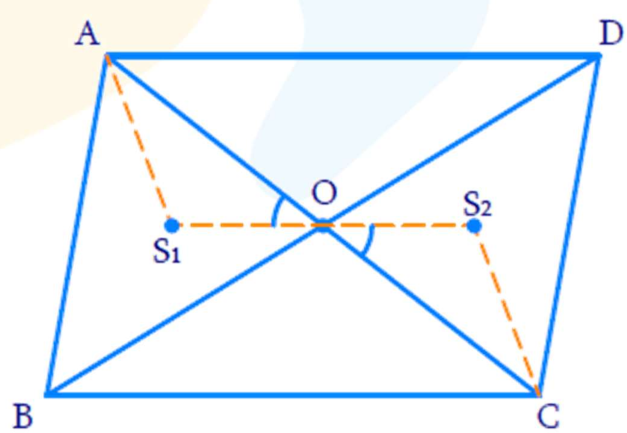


S6. Recall that the orthocenter of a triangle is the point of concurrence of its altitudes. Consider the following figure:



To show that H is the orthocenter of ΔABC, we will show that $AH \perp BC$. From there, it will follow similarly that $BH \perp CA$, and $CH \perp AB$, which means that the altitudes are concurrent at H, i.e., H is the orthocenter.
Note that O_1CO_3H and O_3BO_3A are both rhombuses (why?), and they share a common side O_3H. In other words, two parallelograms, O_1CO_3H and O_2BO_3H share a common side O_3H.

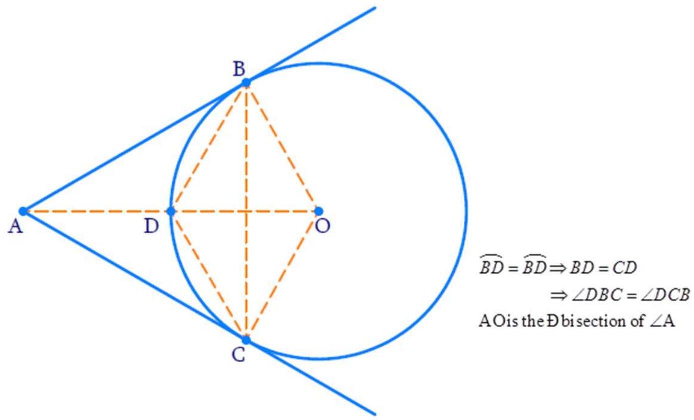
S7. Two circles will touch each other if the distance between their centers is the same as the sum of their radii. In the following figure, S_1 and S_2 are the circumcenters of ΔAOB and ΔCOD respectively. If we are able to show that S_1, O and S_2 are collinear, this would imply that $S_1S_2 = S_1O + OS_2$, that is the distance between the centers of the circumcircles of the two triangles is equal to the sum of their radii, that is, the two circles touch each other:



Since $\Delta OAB \cong \Delta OCD$ (why?), their circumcircles will have equal radii. This means that S_1O and S_1A are equal to S_2O and S_2O. Clearly, $\Delta OS_1A \cong \Delta OS_2C$ (by the SSS criterion). Thus,

$\angle AOS_1 = \angle COS_2$. Since AOC is a straight line, this equality of angles means that S_1OS_2 is also a straight line. This completes our proof.

S8. The incenter is the point of concurrence of the angle bisectors in a triangle. Consider the following figure, and note that A, D and O must be collinear (why?):



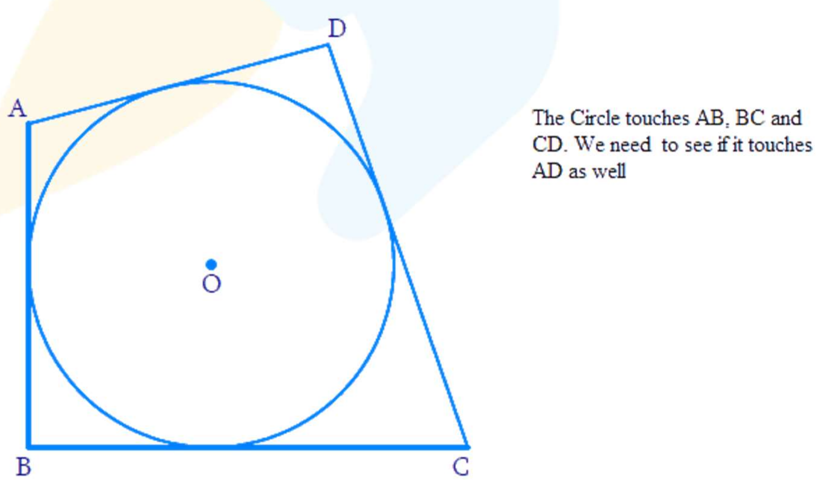
Using the alternate segment theorem and the observations made in the figure, we have:

$$\angle ABD = \angle DCB = \angle DBC$$

$$\Rightarrow \quad \angle ABD = \angle DBC$$

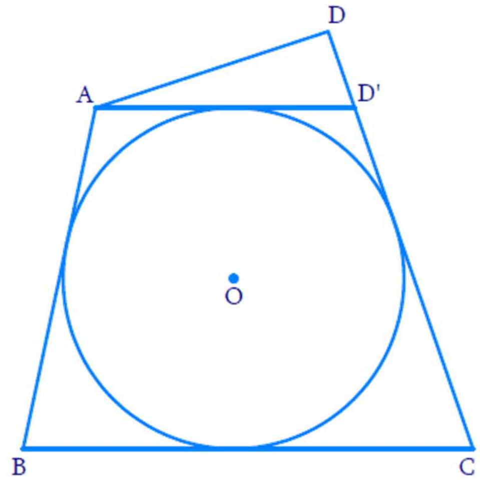
Thus, BD is the angle bisector of $\angle ABC$. Similarly, CD is the angle bisector of $\angle ACB$. We already know that AO (hence AD) is the angle bisector of $\angle A$. Clearly, D is the incenter of ΔABC.

S9. Consider the following figure, in which we have drawn a circle touching the sides AB, BC and CD of the quadrilateral. This can always be done. For example, if you determine the point of intersection O of $\angle B$ and $\angle C$, you will have determined a point which is equidistant from AB, BC and CD. Thus, a circle with center O can be drawn to touch each of these three sides:



We will now show that given the fact that AB + CD = AD + BC, this circle must also touch AD.

Assume that this is not the case. Through A, draw AD' tangent to the circle which intersects CD (or CD extended) at D':



We have shown earlier that if a circle can be inscribed inside a quadrilateral; the sum of one pair of opposite sides is equal to the sum of the other pair of opposite sides. Thus,

$$AB + CD' = AD' + BC$$

$$\Rightarrow AB + (CD - DD') = AD' + BC$$

$$\Rightarrow (AB + CD) - DD' = AD' + BC$$

$$\Rightarrow AD + BC - DD' = AD' + BC \ (\text{given})$$

$$\Rightarrow AD = AD' + DD'$$

However, this relation violates the triangle inequality for $\Delta ADD'$ (the sum of two sides cannot be equal to the third side). Thus, our original assumption that AD is not tangent to the circle is incorrect. We conclude that the circle touches AD.

Note: There is a slight problem with the proof above. It is the implicit assumption that the circle will touch AD somewhere between A and D. But think what would happen if the quadrilateral is non-convex.

S10. Hints: (a) Draw any two chords of the circle and construct their perpendicular bisectors. The center of the circle will be the point of intersection of these perpendicular bisectors.

(b) Use the alternate segment theorem. First, through P, draw chords PQ and PR. Join QR. Suppose that PQ subtends an angle of θ at R. Draw angle QPS equal to θ at the side opposite to point R. Extend PS on both sides. By virtue of the alternate segment theorem, PS (extended) will be the tangent to the circle at P.

(c) (i) Let O be the center of the circle. Using PO as diameter, draw a circle. Suppose that this circle intersects the original circle at A and B. Then, PA and PB are the tangents from P to the circle. Can you see how? (ii) This is slightly more difficult. Through P, draw a secant to the circle, intersecting it at A and B (A being the nearer point of intersection). Draw PC = PA on this secant such that C and A are on the opposite sides of P. Now, with BC as diameter, draw a semi-circle. Next, through P, draw a perpendicular to BC intersecting this semi-circle in D. Now, you can prove that $PD^2 = PC \times PB = PA \times PB$. Also, if PT is a tangent from P to the original circle, then PT will satisfy the relation $PT^2 = PA \times PB$. This means that PD = PT. Thus, we have found the length of the tangent from P to the circle. Now, simply draw a circle of radius PD intersecting the circle in T and T'. Join PT and PT' – these are the required tangents.

(d) Self-exercise.