Numbers and Number Systems

Review Exercise Answers
Level-1

Single Choice Correct Only

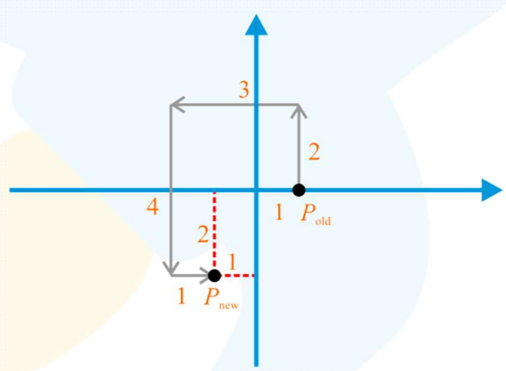
S1. (A), (B), (D) and (E). To check whether each number lies within the specified interval or not, you can mentally calculate the approximate decimal representation of that number, or you can compare the relative magnitude of the numerator with the denominator. For example, if you consider $\frac{11}{8}$, you can see that 11 is larger than 8, but not as large as twice of 8. Therefore, $\frac{11}{8}$ must be greater than 1 but less than 2.

S2. The correct options are (A), (C) and (D). The numbers in options (B) and (D) are rational:

$$\sqrt[6]{64} = 2$$
$$\frac{1}{\sqrt[4]{81}} = \frac{1}{3}$$

S3. The correct option is (A). The real part of his new position is 2, and the non-real part of his new position is 3 times iota, and so his new position can be defined as $P_{new} = 2 + 3i$.

S4. The correct option is (B). The path covered by Beta is shown in the figure below:



Clearly, his new position (from the origin) is one unit west and two units south, and so in complex form, $P_{new} = -1 - 2i$.

S5. (A). We have:

$$S = \frac{2}{7+\sqrt{5}} \times \frac{7-\sqrt{5}}{7-\sqrt{5}} = \frac{2\left(7-\sqrt{5}\right)}{49-5}$$
$$= \frac{7-\sqrt{5}}{22}$$

S6. (B). We have:

$$S = \frac{3}{\sqrt{13}+\sqrt{7}} \times \frac{\sqrt{13}-\sqrt{7}}{\sqrt{13}-\sqrt{7}} = \frac{3\left(\sqrt{13}-\sqrt{7}\right)}{13-7}$$

$$= \frac{\sqrt{13}-\sqrt{7}}{2}$$

S7. (B). Self-exercise.

S8. (D). Since the set has the element 0, it is not contained in $\mathbb{N}$, but it is contained in W. Note that since it is a finite set, it is a subset of W and not the same as W.

S9. (C). Division by 0 is an invalid mathematical operation in every number system, and therefore q can never be equal to 0. In higher mathematics, you will encounter the concept of limits, using which we can talk about q approaching the value of 0, and as q comes closer and closer to 0, the number $\frac{p}{q}$ keeps on increasing in magnitude. But even then, q can never be *exactly* 0.

S10. (C). The sum and the product of two integers is again an integer and therefore options (A) and (C) are true. Also, the product of two integers can be equal to 0 only when one of them is 0. Dividing two integers may not yield an integer, and hence (C) is incorrect.

S11. (B). Let

$$x = 23.4\overline{26}$$

$$\Rightarrow \quad x = 23.42626... \quad \text{(i)}$$

Multiplying equation (i) by 10, we get

$$10x = 234.2626... \qquad \text{(ii)}$$

Multiplying equation (ii) by 100, we get

$$1000x = 23426.26 \qquad ...\text{(iii)}$$

Subtracting equation (ii) from (iii), we get

$$990x = 23192$$

$$\Rightarrow \quad x = \frac{23192}{990}$$

S12. (B). We have:

$$\sqrt{121} = \sqrt{11\times 11} = 11$$

$$\sqrt[4]{\frac{81}{16}} = \sqrt[4]{\frac{3\times 3\times 3\times 3}{2\times 2\times 2\times 2}} = \frac{3}{2}$$

Thus, the numbers $\sqrt{121}$ and $\sqrt[4]{\frac{81}{16}}$ are rational. The last number is also rational:

$$
\begin{aligned}
& x = 0.\overline{27413} = 0.2741327413... \\
\Rightarrow \quad & 100000x = 27413.2741327413 \\
\Rightarrow \quad & 99999x = 27413 \\
\Rightarrow \quad & x = \frac{27413}{99999}
\end{aligned}
$$

π is irrational, as is the fourth root of 2 (why?). The fifth power of 10000 is also irrational. The number 10000 is the fourth power of 10, that is, $10000 = 10^4$. If instead of 10000, we were given the number 100000, then the fifth power would have been rational (equal to 10).

S13. (C). The lengths of the diagonals can be calculated for each rectangle using the Pythagoras theorem:

$$
\begin{aligned}
& P : \sqrt{2^2 + 1^2} = \sqrt{5} \\
& Q : \sqrt{3^2 + 2^2} = \sqrt{13} \\
& R : \sqrt{4^2 + 3^2} = \sqrt{25} = 5 \\
& S : \sqrt{5^2 + 4^2} = \sqrt{41}
\end{aligned}
$$

We see that the diagonals of *P*, *Q* and *S* have irrational lengths, while the diagonal of *R* has a rational length.

S14. (C). We have:

$$
\begin{aligned}
& BC = \sqrt{\left(\sqrt{10}\right)^2 - \left(2\sqrt{2}\right)^2} = \sqrt{10 - 8} = \sqrt{2} \\
\Rightarrow \quad & l = AB + BC + CA = 2\sqrt{2} + \sqrt{2} + \sqrt{10} \\
& \quad = 3\sqrt{2} + \sqrt{10} \\
& \text{and } A = \frac{1}{2} \times AB \times BC = \frac{1}{2} \times 2\sqrt{2} \times \sqrt{2} = 2
\end{aligned}
$$

Thus, it is clear that *l* is irrational while *A* is rational. Can you prove that *l* is irrational.

S15. (A), because the two terms will separately have terminating representations, and so will their sum.

S16. (D). We have:

(A) $\sqrt{2\frac{7}{9}}=\sqrt{\frac{25}{9}}=\frac{5}{3}$

(B) $\sqrt{1-\frac{32}{81}}=\sqrt{\frac{81-32}{81}}=\sqrt{\frac{49}{81}}=\frac{7}{9}$

(C) $\sqrt{\frac{27}{243}}=\sqrt{\frac{1}{9}}=\frac{1}{3}$

(D) $\sqrt{3\frac{1}{4}}=\sqrt{\frac{13}{4}}=\frac{\sqrt{13}}{2}$

The numbers in the options (A), (B) and (C) are rational, while the number in option (D) is irrational.

S17. (D). We will rationalize the denominator of the given expression using the following algebraic identity:

$$(a-b)(a^2+ab+b^2)=a^3-b^3$$

The rationalizing factor for the denominator will thus be

$$\left(\sqrt[3]{3}\right)^2+\left(\sqrt[3]{3}\right)\left(\sqrt[3]{2}\right)+\left(\sqrt[3]{2}\right)^2$$
$$=\sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4}$$

Using this to rationalize the denominator, we have:

$$\frac{1}{\sqrt[3]{3}-\sqrt[3]{2}}\times\frac{\sqrt[3]{4}+\sqrt[3]{6}+\sqrt[3]{9}}{\sqrt[3]{4}+\sqrt[3]{6}+\sqrt[3]{9}}$$
$$=\frac{\sqrt[3]{4}+\sqrt[3]{6}+\sqrt[3]{9}}{\left(\sqrt[3]{3}\right)^3-\left(\sqrt[3]{2}\right)^3}=\frac{\sqrt[3]{4}+\sqrt[3]{6}+\sqrt[3]{9}}{3-2}$$
$$=\sqrt[3]{4}+\sqrt[3]{6}+\sqrt[3]{9}$$

S18. (B). The rationalizing factor for the denominator will thus be

$$\left(\sqrt[3]{5}\right)^2+\left(\sqrt[3]{5}\right)\left(\sqrt[3]{2}\right)+\left(\sqrt[3]{2}\right)^2$$
$$=\sqrt[3]{25}+\sqrt[3]{10}+\sqrt[3]{4}$$

Multiplying this with the numerator and the denominator of the original expression, we will have:

$$\frac{3\left(\sqrt[3]{4}+\sqrt[3]{10}+\sqrt[3]{25}\right)}{\left(\sqrt[3]{5}\right)^3-\left(\sqrt[3]{2}\right)^3}=\frac{3\left(\sqrt[3]{4}+\sqrt[3]{10}+\sqrt[3]{25}\right)}{3}$$
$$=\sqrt[3]{4}+\sqrt[3]{10}+\sqrt[3]{25}$$

S19. (B). We have

$$\frac{1}{\left(\sqrt{7}+3\right)^3}+\frac{1}{\left(\sqrt{7}-3\right)^3}$$

$$=\frac{\left(\sqrt{7}-3\right)^3+\left(\sqrt{7}+3\right)^3}{\left(\sqrt{7}+3\right)^3\left(\sqrt{7}-3\right)^3}$$

$$=\frac{14\sqrt{7}+54\sqrt{7}}{(-2)^3}\ \text{(how?)}$$

$$=\frac{68\sqrt{7}}{-8}=-\frac{17\sqrt{7}}{2}$$

S20. (B). On multiplying the numerator and the denominator by $\sqrt[4]{3}$, we get

$$\frac{3}{5\sqrt[4]{27}}=\frac{3}{5\sqrt[4]{27}}\times\frac{\sqrt[4]{3}}{\sqrt[4]{3}}$$

$$=\frac{3\sqrt[4]{3}}{5\sqrt[4]{27\times 3}}=\frac{3\sqrt[4]{3}}{5\sqrt[4]{3^4}}$$

$$=\frac{3\sqrt[4]{3}}{5\times 3}=\frac{\sqrt[4]{3}}{5}$$

One or More than Once Choices Correct

S21. (B) and (D). The collection of non-negative integers is the set {0, 1, 2, 3,…}, which can be described both as the set of Whole Numbers and the set of all except the negative Integers. This set will be superset of the set of Natural Numbers, which is the same as the set of Positive Integers.

S22. (B) and (D). The set of naturals is a subset of the set of whole numbers, which is a subset of the set of integers, which is further a subset of the set of rational numbers. Thus, every natural number is a whole number, every whole number is an integer, and every integer is a rational number.

S23. (B), (C), (D) and (E). Self-exercise.

S24. (A) and (D). For a valid rational number p/q, p can be any integer, but q must be a non-zero integer, otherwise the expression p/q becomes mathematically undefined, because division by zero makes no mathematical sense.

S25. (A), (C) and (D). Explanations:

- **(A) True**. Adding two natural numbers will always yield a larger natural number.
- **(B) False**. Subtracting two whole numbers can yield negative integers, which are not whole numbers.
- **(C) True**. Every integer is a rational number.
- **(D) True**. The integer set is **closed** under the addition (as well as subtraction) operation, which means that adding or subtracting two integers will always yield an integer.
- **(E) False**. Between any two different numbers, there will be infinitely many rational numbers.

S26. The only correct option is (D). In the Rational set, no matter how close a rational you find to 1, you can still find infinitely many rational numbers between 1 and that number.

S27. (A) and (C). It should be obvious that (A) is correct and (B) is incorrect. (C) is correct because any number in the set W or $\mathbb{N}$ can be written as a rational number. This means that every element of W and $\mathbb{N}$ lies in $\mathbb{Q}$. Thus, $\mathbb{Q}$ contains both W as well as $\mathbb{N}$. (D) is incorrect because a rational number may not be an integer, but every integer is a rational number. Thus, every element of $\mathbb{Z}$ lies in $\mathbb{Q}$, which means that $\mathbb{Q}$ contains $\mathbb{Z}$, and not the other way around.

S28. (A) and (C). All the four expressions represent the same number. These are all different forms in which the number can be written, and the simplest form among all these would be $\frac{1}{2}$. When you cancel out the common factors (if any), you are essentially changing the form of the number to a simpler one, but you are not changing the number itself. The number is the same, and correspondingly, there will be just one point on the number line for all these four forms. Therefore, the other options (B), (D) and (E) are incorrect.

S29. (A), (B), (C) and (E). Point-wise explanations:

(A) CORRECT. There are infinitely many integers. No matter how large an integer you select, there will always exist infinitely many integers which are larger than that integer.

(B) CORRECT. You can represent multiples of 41 by $41k$, where k is a natural number. Obviously, since there are infinitely many natural numbers, there will also be infinitely many numbers of the form $41k$, or infinitely many multiples of 41.

(C) CORRECT. The rational numbers $\frac{1}{1000}$ and $\frac{1}{1001}$ are extremely close to each other. If you were to plot them accurately on a number line on a page (on a cm scale), your eyes wouldn't be able to differentiate between the two points. Even then, there are an infinite number of rational numbers between these two numbers. No matter how close two rational numbers are to each other, there will always exist infinitely many rational numbers between those two rationals.

(D) INCORRECT. The two numbers $\frac{1}{1001}$ and $\frac{10000}{1001}$ are finite, and so there will exist only a finite number of integers between the two.

(E) CORRECT. By "negative rational numbers greater than -1", what is meant is all the rational numbers between -1 and 0. There will be infinitely many such rationals.

S30. (A) and (D). Between any two unequal (finite) rational numbers, there will obviously be a finite number of integers, since the difference between the two rational numbers will be finite. On the other hand, there will be infinitely many rational numbers between the two given rational numbers, no matter how close they are to each other.

S31. (A) and (C), because the denominators of these rationals have only 2 and 5 as prime factors. (D) is anyway irrational.

S32. (B), (C) and (D), as the denominators in these rationals have prime factors other than 2 and 5.

S33. (B). We have:

$$l = 2(\pi)\left(\frac{1}{\pi}\right) = 2,$$

which is rational, whereas,

$$A = (\pi)\left(\frac{1}{\pi}\right)^2 = \frac{1}{\pi},$$

which is irrational, as π itself is irrational.

S34. (A), (C) and (D). We have:

$$\sqrt[3]{8} = \sqrt[3]{2\times2\times2} = 2$$
$$\sqrt[3]{27} = \sqrt[3]{3\times3\times3} = 3$$
$$\sqrt[3]{64} = \sqrt[3]{4\times4\times4} = 4$$

The cube-roots of 9 and 81 are irrational. 9 is the square of 3, while 81 is the fourth power of 3. Therefore,

$$\sqrt[3]{9} = \sqrt[3]{3^2} = 3^{\frac{2}{3}}$$
$$\sqrt[3]{81} = \sqrt[3]{3^4} = 3^{\frac{4}{3}}$$

S35. (C), (D) and (E). In $\mathbb{N}$ and $\mathbb{Z}$, for any given number, the next number is well defined. However, in the other three sets, for any given number, there is no next number, since you can find a number as close to the given number as you want to.

Integer Answers

S36. The answer is 1. We have:

$$\left(\sqrt{5}+2\right)\left(\sqrt{5}-2\right) = \left(\sqrt{5}\right)^2 - (2)^2$$
$$= 5 - 4 = 1$$

S37. The correct answer is 0. We have: $\left[\frac{34}{5}\right] = [6.8]$. The greatest integer which is less than 6.8 is 6. Therefore, we have:

$$\left[\frac{34}{5}\right] = [6.8] = 6, \quad [6] = 6$$
$$\Rightarrow \left[\frac{34}{5}\right] - [6] = 6 - 6 = 0$$

S38. The correct answer is -1. We have:

$$\left[-\frac{21}{4}\right] = [-5.25]$$

The greatest integer which is less than -5.25 is -6. Therefore, we have:

$$\left[-\frac{21}{4}\right]=[-5.25]=-6$$

Also, the greatest integer which is less than 5.25 is 5, and so:

$$\left[-\frac{21}{4}\right]+\left[\frac{21}{4}\right]=[-5.25]+[5.25]$$
$$=-6+5=-1$$

S39. We have:

$$S=\frac{5+7\sqrt{3}}{2-11\sqrt{3}}\times\frac{2+11\sqrt{3}}{2+11\sqrt{3}}$$
$$=\frac{10+55\sqrt{3}+14\sqrt{3}+77\times 3}{4-\left(11\sqrt{3}\right)^2}$$
$$=\frac{241+69\sqrt{3}}{4-363}=-\left(\frac{241+69\sqrt{3}}{359}\right)$$

Thus,

$$b=359, a=241$$
$$\Rightarrow b-a=118$$

S40. The answer is 70. We have:

$$S=\frac{10}{\sqrt[3]{7}+\sqrt[3]{3}}\times\frac{\sqrt[3]{7^2}-\sqrt[3]{21}+\sqrt[3]{3^2}}{\sqrt[3]{7^2}-\sqrt[3]{21}+\sqrt[3]{3^2}}$$
$$=\frac{10\left(\sqrt[3]{49}-\sqrt[3]{21}+\sqrt[3]{9}\right)}{7+3}$$
$$=\sqrt[3]{49}+\sqrt[3]{9}-\sqrt[3]{21}$$

Thus,

$$a=49, b=21$$
$$\Rightarrow a+b=70$$

S41. The correct answer is 19. We have:

$$\{4.7\}=0.7,\ \left\{\frac{1}{4}\right\}=\{0.25\}=0.25$$
$$\Rightarrow \{4.7\}+\left\{\frac{1}{4}\right\}=0.7+0.25=0.95$$
$$=\frac{95}{100}=\frac{19}{20}$$
$$\Rightarrow 20\left(\{4.7\}+\left\{\frac{1}{4}\right\}\right)=19$$

S42. The correct answer is 1. We have:

$$\left\{\frac{999}{100}\right\} = \{9.99\} = 0.99$$
$$\left\{\frac{1}{100}\right\} = \{0.01\} = 0.01$$
$$\Rightarrow \left\{\frac{999}{100}\right\} + \left\{\frac{1}{100}\right\} = 0.99 + 0.01$$
$$= 1.00 = 1$$

S43. We have:

$$9\left(\frac{2}{3} + 0.\overline{11}\right) = 9\left(\frac{2}{3} + \frac{11}{99}\right)$$
$$= 9\left(\frac{2}{3} + \frac{1}{9}\right) = 9\left(\frac{6+1}{9}\right) = 7$$

S44. We have

$$\left(a + b\sqrt{2}\right)^2 = 99 + 70\sqrt{2}$$
$$\Rightarrow a^2 + \left(b\sqrt{2}\right)^2 + 2a\left(b\sqrt{2}\right) = 99 + 70\sqrt{2}$$
$$\Rightarrow \left(a^2 + 2b^2\right) + \left(2ab\right)\sqrt{2} = 99 + 70\sqrt{2}$$

Now, we note that since *a* and *b* are integers, the only way the two sides above can be equal is if

$$a^2 + 2b^2 = 99$$
$$2ab = 70 \Rightarrow ab = 35$$

By a little bit of hit-and-trial, we can conclude that *a* is equal to 7 and *b* is equal to 5, and so the required value is $a + b = 12$.

S45. We rationalize the denominator of the given expression:

$$\frac{3+2\sqrt{3}}{5-2\sqrt{3}} \times \frac{5+2\sqrt{3}}{5+2\sqrt{3}} = \frac{15+6\sqrt{3}+10\sqrt{3}+12}{25-12}$$
$$= \frac{27+16\sqrt{3}}{13} = \frac{27}{13} + \frac{16}{13}\sqrt{3}$$
$$\Rightarrow \; a = \frac{27}{13}, \, b = \frac{16}{13}$$
$$\Rightarrow \; a - b = \frac{11}{13} \; \Rightarrow \; 13\left(a-b\right) = 11$$

S46. On squaring both sides, we get[/j][-t]

$$\left(\sqrt{12-x\sqrt{5}}\right)^2=\left(\sqrt{10}-\sqrt{2}\right)^2$$
$$\Rightarrow\ 12-x\sqrt{5}=\left(\sqrt{10}\right)^2+\left(\sqrt{2}\right)^2-2\sqrt{10}\sqrt{2}$$
$$\Rightarrow\ 12-x\sqrt{5}=10+2-2\sqrt{20}$$
$$\Rightarrow\ -x\sqrt{5}=-4\sqrt{5}$$
$$\Rightarrow\ x=4$$

S47. We have:

$$b=\frac{1}{a}=\frac{1}{5+2\sqrt{6}}$$
$$=\frac{1}{5+2\sqrt{6}}\times\frac{5-2\sqrt{6}}{5-2\sqrt{6}}$$
$$=\frac{5-2\sqrt{6}}{(5)^2-\left(2\sqrt{6}\right)^2}$$
$$=\frac{5-2\sqrt{6}}{25-24}$$
$$=5-2\sqrt{6}$$

Thus, we have:

$$a+b=5+2\sqrt{6}+5-2\sqrt{6}$$
$$\Rightarrow\ a+b=10$$

On squaring both sides, we get

$$(a+b)^2=(10)^2$$
$$\Rightarrow\ a^2+b^2+2ab=100$$
$$\Rightarrow\ a^2+b^2+2=100\quad\left[\because\ a=\frac{1}{b}\right]$$
$$\Rightarrow\ a^2+b^2=98$$

Assertion-Reasoning

S48. (E). Both (A) and (R) are false. (A) is false because there are lengths which cannot be represented by rational numbers. These correspond to the gaps or holes on the rational number line. (R) is false because a number of the form $\frac{p}{q}$ is rational only if p and q are both integers (and q is non-zero, of course).

There is no link between (A) and (R).

S49. (E). Both (A) and (R) are false. Every irrational number is *not* a surd. There are irrational numbers which cannot be expressed at roots of rational numbers.
However, we do once again note that every surd is an irrational number.

Matrix-Match

S50. (a) to (r), (b) to (s), (c) to (q) and (d) to (p).

$$x = 0.4 = \frac{4}{10} = \frac{2}{5}$$
$$x = 0.333... \Rightarrow 10x = 3.333...$$
$$\Rightarrow 9x = (3.333...) - (0.333...) = 3$$
$$\Rightarrow x = \frac{3}{9} = \frac{1}{3}$$
$$x = 0.1666... \Rightarrow \begin{cases} 10x = 1.666... \\ 100x = 16.666... \end{cases}$$
$$\Rightarrow 100x - 10x = 90x = 16 - 1 = 15$$
$$\Rightarrow x = \frac{15}{90} = \frac{1}{6}$$
$$x = 0.\overline{285714} \Rightarrow 1000000x = 285714.\overline{285714}$$
$$\Rightarrow 999999x = 285714 \Rightarrow x = \frac{285714}{999999} = \frac{2}{7}$$

Miscellaneous

S51. The given statement is false. There is no restriction on what values p can take, but even if p is 0, q can still not be equal to 0. The expression $\frac{0}{0}$ is mathematically undefined. However, if q is not zero, then $\frac{0}{q}$ is defined and is equal to 0.

S52. The given statement is true, since the fractional part is by definition the difference between the number and its greatest integer part:

$$\{x\} = x - [x]$$
$$\Rightarrow x - \{x\} = [x]$$

S53. Either both are positive or both are negative, i.e., both have the same sign.

S54. Multiplying a negative number on both sides of an inequality changes the *direction* of the inequality. Thus, $pr > qr$.

S55. The given statement is false. The value $\frac{22}{7}$ is only an approximation of π. As we have discussed, π is an irrational number whose value cannot be represented in a rational form. Always keep this fact in mind: π is only approximately equal to $\frac{22}{7}$. An even better rational approximation to π is $\frac{355}{113}$ but once again, this is only an approximation. We generally use the value $\frac{22}{7}$ for π because it is convenient to work with numerically.

S56. The set of Rationals is countable (along with its subsets). However, the set of Irrationals and the set of Reals are uncountable sets. The proof that the set of rationals is countable, is very interesting; you are urged to research this topic. Technically, we say that the set of Rationals is countably infinite (like the set of Naturals, Whole Numbers or Integers).